Central extensions of restricted Lie superalgebras and classification of p-nilpotent Lie superalgebras in dimension 4

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Outline

Let $\mathbb{K}$ be a field of characteristic $p > 2$, algebraically closed.

- **Our goals:**
  - classification of low-dimensional *p-nilpotent restricted* Lie superalgebras over $\mathbb{K}$.
  - superization of formulas for the restricted cohomology.

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Let L be a p-nilpotent restricted Lie superalgebra of dimension n. Then, L is isomorphic to a central extension by a restricted 2-cocycle of a p-nilpotent restricted Lie superalgebra of dimension $n-1$.

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- **What do we need?** Restricted 2-cocycles of the restricted cohomology for restricted Lie superalgebras.

Restricted Lie algebras

Let $\mathbb{K}$ a field of characteristic $p > 2$ and A an associative $\mathbb{K}$-algebra. With the commutator, it's a Lie algebra. The adjoint representation is then given by

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Then, if $m = p$, we obtain

$$\mathrm{ad}_x^p(y) = x^p y - y x^p = \mathrm{ad}_{x^p}(y).$$

Restricted Lie algebras

Definition (Jacobson)

A **restricted Lie algebra** *is a Lie algebra L equipped with a map $(\cdot)^{[p]} : L \longrightarrow L$ satisfying for all $x, y \in L$ and for all $\lambda \in \mathbb{K}$:*

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2. $\left[x, y^{[p]}\right] = [[\cdots[x, \overbrace{y], y], \cdots, y}^{p \text{ terms}}]$;
3. $(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x, y)$,

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with $is_i(x, y)$ the coefficient of Z^{i-1} in $\mathrm{ad}^{p-1}_{Zx+y}(x)$. Such a map $(-)^{[p]} : L \longrightarrow L$ is called p-map.

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Example: any associative algebra A with $[a, b] = ab - ba$ and $a^{[p]} = a^p$, $\forall a, b \in A$.

Restricted Lie algebras

Very useful :

$$\sum_{i=1}^{p-1} s_i(x,y) = \sum_{\substack{x_i = x \text{ or } y \\ x_p = x,\ x_{p-1} = y}} \frac{1}{\sharp\{x\}} [x_1, [x_2, [\dots, [x_{p-1}, x_p]\dots],$$

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Definition

A Lie algebra morphism $f : \left(L, [\cdot,\cdot], (\cdot)^{[p]}\right) \to \left(L', [\cdot,\cdot]', (\cdot)^{[p]'}\right)$ *is called* **restricted** *if*

$$f\left(x^{[p]}\right) = f(x)^{[p]'},\ \forall x \in L.$$

A L*-module* M *is called* **restricted** *if*

$$x^{[p]} \cdot m = \Big(\overbrace{x \cdot (x \cdots (x}^{p \text{ terms}} \cdot m) \cdots)\Big),\ \forall x \in L,\ \forall m \in M.$$

Lie superalgebras

Definition

A Lie superalgebra $L = L_{\bar{0}} \oplus L_{\bar{1}}$ is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space equipped with a bilinear map $[\cdot,\cdot] : L \times L \to L$ satisfying for $x, y, z \in L$:

1. $|[x,y]| = |x| + |y|$;
2. $[x,y] = -(-1)^{|x||y|}[y,x]$;
3. $(-1)^{|x||z|}[x,[y,z]] + (-1)^{|x||y|}[y,[z,x]] + (-1)^{|y||z|}[z,[x,y]] = 0.$

If $p = 3$, the identity $[x,[x,x]] = 0, \ x \in L_{\bar{1}}$ has to be added as an axiom as well.

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Let $f : V \to W$ be a map between $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces. Then:

- the map f is called **even** if $f(V_{\bar{i}}) \subset W_{\bar{i}}$;
- the map f is called **odd** if $f(V_{\bar{i}}) \subset W_{\overline{i+1}}$;

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A **restricted Lie superalgebra** *is a Lie superalgebra* $L = L_{\bar{0}} \oplus L_{\bar{1}}$ *such that*

1. *The even part* $L_{\bar{0}}$ *is a restricted Lie algebra;*
2. *The odd part* $L_{\bar{1}}$ *is a Lie* $L_{\bar{0}}$*-module;*
3. $[x, y^{[p]}] = [[\ldots[x, \overbrace{y], y], \ldots, y}^{p\ \text{terms}}], \ \forall x \in L_{\bar{1}}, \ y \in L_{\bar{0}}.$

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We can define a map $(\cdot)^{[2p]} : L_{\bar{1}} \to L_{\bar{0}}$ *by*

$$x^{[2p]} = \left(x^2\right)^{[p]}, \ \textit{with } x^2 = \frac{1}{2}[x, x], \ x \in L_{\bar{1}}.$$

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Theorem (Jacobson)

Let $(e_j)_{j \in J}$ *be a basis of* $L_{\bar{0}}$*, and let the elements* $f_j \in L_{\bar{0}}$ *be such that* $(\mathrm{ad}_{e_j})^p = \mathrm{ad}_{f_j}$*. Then, there exists exactly one* $p|2p$*-mapping* $(\cdot)^{[p|2p]} : L \to L$ *such that*

$$e_j^{[p]} = f_j \quad \textit{for all } j \in J.$$

Restricted p-nilpotent Lie superalgebras

Let L be a Lie superalgebra. We define a *descending central sequence* by

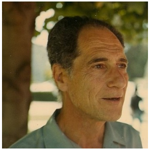
$$C^0(L) = L, \quad \text{and} \quad C^{k+1}(L) = \left[C^k(L), L\right].$$

The Lie superalgebra L is called *nilpotent* if there exists $k \geq 0$ such that $C^k(L) = 0$.

Suppose that L is restricted. Then L is called *p-nilpotent* if there exists $n \geq 0$ such that $x^{[p]^n} = 0 \ \forall x \in L_{\bar{0}}$. Any p-nilpotent restricted Lie superalgebra is nilpotent.

A (very) brief history of restricted cohomology

- 1955 (Hochschild): $H^n_*(L, M) := \mathrm{Ext}^n_{U(L)}(\mathbb{F}, M)$.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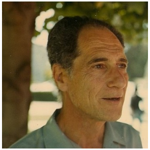

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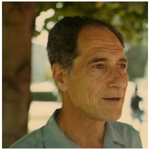

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- 2020: attempt to generalize to the superalgebras case.

Restricted cohomology for restricted Lie superalgebras

Let $L = L_{\bar{0}} \oplus L_{\bar{1}}$ be a restricted Lie superalgebra and let M be a L-supermodule.

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Definition (Restricted 2-cochains)

*Let $\varphi \in C^2_{CE}(L, M)$ (ordinary Chevalley-Eilenberg 2-cochain) and $\omega : L \longrightarrow M$. Then ω is φ-**compatible** if*

1. $\omega(\lambda x) = \lambda^p \omega(x), \; \lambda \in \mathbb{F}, \; x \in L_{\bar{0}};$

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$$\omega(x) + \omega(y) + \sum_{\substack{x_i = x \text{ or } y \\ x_1 = x, \ x_2 = y}} \frac{1}{\pi(x)} \sum_{k=0}^{p-2} (-1)^k x_p \dots x_{p-k+1} \varphi\big([[\dots[x_1, x_2], x_3]\dots, x_{p-k-1}], x_{p-k}\big),$$

with $x, y \in L$, $\pi(x)$ the number of factors x_i equal to x.

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$$C^2_*(L,M) = \left\{(\varphi,\omega),\ \varphi \in C^2_{CE}(L,M),\ \omega \text{ is } \varphi\text{-compatible}\right\}$$

$\rightsquigarrow$ We have a similar (although more complicated) definition for $C^3_*(L,M)$.

Restricted cohomology for restricted Lie superalgebras

- A **restricted 2-cocycle** is an element $(\alpha, \beta) \in C_*^2(L, M)$ such that

  1. $(-1)^{|x||z|}\alpha(x, [y, z]) + (-1)^{|y||x|}\alpha(y, [z, x]) + (-1)^{|z||y|}\alpha(z, [x, y]) = 0$, $\forall x, y, z \in L$;
  2. $$\alpha\left(x, y^{[p]}\right) - \sum_{i+j=p-1} (-1)^i y^i \alpha\Big([x, \underbrace{y, \cdots, y}_{j \text{ terms}}], y\Big) + (-1)^{|x||\alpha|} x\beta(y) = 0,$$ $\forall x \in L,\ y \in L_{\bar{0}}$.

  The space of restricted 2-cocycle is denoted by $Z_*^2(L, M)$.

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- A **restricted 2-coboundary** is an element $(\alpha, \beta) \in C_*^2(L, M)$ such that $\exists \varphi \in \mathrm{Hom}(L, M)$,

  1. $\alpha(x, y) = \varphi\big([x, y]\big) - (-1)^{|x||\varphi|} x\varphi(y) + (-1)^{|y|(|\varphi|+|x|)} y\varphi(x),\ \forall x, y \in L$;
  2. $\beta(x) = \varphi\left(x^{[p]}\right) - x^{p-1}\varphi(x), \forall x \in L_{\bar{0}}$.

  The space of restricted 2-coboundaries is denoted by $B_*^2(L, M)$.

Restricted cohomology for restricted Lie superalgebras

The previous formulae define maps

$$0 \longrightarrow C^0_*(L, M) \xrightarrow{d^0_*} C^1_*(L, M) \xrightarrow{d^1_*} C^2_*(L, M) \xrightarrow{d^2_*} C^3_*(L, M),$$

with $d^0_* = d^0_{CE}$.

Theorem

We have $d^2_* \circ d^1_* = 0$. Therefore, the quotient space

$$H^2_*(L; M) = Z^2_*(L; M)/B^2_*(L; M)$$

is well defined.

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Let L be a restricted Lie superalgebra and M a restricted L-module. We define a subspace $C^2_*(L;M)^+ \subset C^2_*(L;M)$ by

$$C^2_*(L;M)^+ := \Big\{(\alpha,\beta) \in C^2_*(L;M),\ \mathrm{Im}(\beta) \subseteq M_{\bar{0}}\Big\}.$$

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Lemma

(i) *We have an inclusion* $B^2_*(L;M)_{\bar{0}} \subset C^2_*(L;M)^+$*.*

(ii) *The space* $C^2_*(L;M)^+$ *is* $\mathbb{Z}_2$*-graded and the degree of an homogeneous element* $(\alpha,\beta) \in C^2_*(L;M)^+$ *is given by* $|(\alpha,\beta)| = |\alpha|$*.*

This Lemma allows us to consider the space $Z^2_*(L;M)^+ := \ker\big(d^2_{*|C^2_*(L;M)^+}\big)$. Thus we can define

$$H^2_*(L;M)^+ := Z^2_*(L;M)^+/B^2_*(L;M)_{\bar{0}}.$$

The space $H^2_*(L;M)^+$ is $\mathbb{Z}_2$-graded.

Central extensions of restricted Lie superalgebras

Let $\left(L, [\cdot,\cdot], (\cdot)^{[p]}\right)$ be a restricted Lie superalgebra, and M be a strongly abelian restricted Lie superalgebra $\left(i.e,\ [m,n] = 0\ \forall m,n \in M,\ \text{and } m^{[p]} = 0\ \forall m \in M_{\bar{0}}\right)$.

A **restricted extension** of L by M is a short exact sequence of restricted Lie superalgebras

$$0 \longrightarrow M \xrightarrow{\iota} E \xrightarrow{\pi} L \longrightarrow 0.$$

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In the case where $\iota(M) \subset \mathfrak{z}(E) := \{a \in E,\ [a,b] = 0\ \forall b \in E\}$, M is a trivial L-module. These extensions are called **restricted central extensions**.

Two restricted central extensions of L by M are called **equivalent** if there is a restricted Lie superalgebras morphism $\sigma : E_1 \to E_2$ such that the following diagram commutes:

$$\begin{tikzcd}
 & & E_1 \arrow[dd, "\sigma"] \arrow[dr, "\pi_1"] & & \\
0 \arrow[r] & M \arrow[ur, "\iota_1"] \arrow[dr, "\iota_2"'] & & L \arrow[r] & 0. \\
 & & E_2 \arrow[ur, "\pi_2"'] & &
\end{tikzcd}$$

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$$0 \longrightarrow M \xrightarrow{\iota} E \xrightarrow{\pi} L \longrightarrow 0.$$

Theorem

Let L be a restricted Lie superalgebra and M a strongly abelian restricted Lie superalgebra. Then, the equivalence classes of restricted central extensions of L by M are classified by $H^2_*(L; M)^+_{\bar{0}}$.

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Structure maps on E. Let $(\varphi, \omega) \in Z^2_*(L; \mathbb{K})^+_{\bar{0}}$. The bracket and the p- map on E are given by

$$[x+m, y+n]_E := [x, y] + \varphi(x, y), \ \forall x, y \in L, \ \forall m, n \in M; \tag{1}$$

$$(x+m)^{[p]_E} := (x)^{[p]} + \omega(x), \ \forall x \in L_{\bar{0}}, \ \forall m \in M_{\bar{0}}. \tag{2}$$

A brief history of classification of restricted Lie algebras

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- 2023 (Maletesta and Siciliano): Classification of p-nilpotent restricted Lie algebras of dimension 5, $p \geq 3$, *using another method* (J. Algebra).

Dimension 3

- sdim$(L) = (1|2)$: $L = \langle e_1 | e_2, e_3 \rangle$.

  1. $\mathbf{L}^1_{1|2} = \langle e_1 | e_2, e_3 \rangle$ (abelian):
     1. $e_1^{[p]} = 0$;
  2. $\mathbf{L}^2_{1|2} = \langle e_1 | e_2, e_3; [e_2, e_3] = e_1 \rangle$:
     1. $e_1^{[p]} = 0$;
  3. $\mathbf{L}^3_{1|2} = \langle e_1 | e_2, e_3; [e_1, e_2] = e_3 \rangle$:
     1. $e_1^{[p]} = 0$.
  4. $\mathbf{L}^4_{1|2} = \langle e_1 | e_2, e_3; [e_3, e_3] = e_1 \rangle$:
     1. $e_1^{[p]} = 0$;

- sdim$(L) = (2|1)$: $L = \langle e_1, e_2 | e_3 \rangle$.

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     1. $e_1^{[p]} = e_2^{[p]} = 0$;
     2. $e_1^{[p]} = e_2, \; e_2^{[p]} = 0$.
  2. $\mathbf{L}^2_{2|1} = \langle e_1, e_2 | e_3; [e_3, e_3] = e_2 \rangle$:
     1. $e_1^{[p]} = e_2^{[p]} = 0$;
     2. $e_1^{[p]} = e_2, \; e_2^{[p]} = 0$.

- sdim$(L) = (3|0)$: $L = \langle e_1, e_2, e_3 \rangle$, (see Schneider-Usefi).

  1. $\mathbf{L}^1_{3|0} = \langle e_1, e_2, e_3 \rangle$ (abelian):
     1. $e_1^{[p]} = e_2^{[p]} = e_3^{[p]} = 0$;
     2. $e_1^{[p]} = e_2, \; e_2^{[p]} = e_3^{[p]} = 0$;
     3. $e_1^{[p]} = e_2, \; e_2^{[p]} = e_3, \; e_3^{[p]} = 0$.
  2. $\mathbf{L}^2_{3|0} = \langle e_1, e_2, e_3; [e_1, e_2] = e_3 \rangle$
     1. $e_1^{[p]} = e_2^{[p]} = e_3^{[p]} = 0$;
     2. $e_1^{[p]} = e_3, \; e_2^{[p]} = e_3^{[p]} = 0$.

Dimension 4: scalar restricted 2-cocycles

Let L be a restricted Lie superalgebra and M be a restricted L-module. Let $(\varphi, \omega) \in Z^2_*(L; M)$ and A be a restricted automorphism of L. An action is given by $A \cdot (\varphi, \omega) := (A\varphi, A\omega)$, with

$$\begin{cases} (A\varphi)(x, y) & = \varphi\big(A(x), A(y)\big), \ \forall x, y \in L \\ (A\omega)(x) & = \omega\big(A(x)\big), \ \forall x \in L_{\bar{0}}. \end{cases} \tag{3}$$

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Proposition

- *Let L be a p-nilpotent restricted Lie superalgebra of dimension n. Then, L is isomorphic to a central extension by a restricted 2-cocycle of a p-nilpotent restricted Lie superalgebra of dimension $n-1$.*
- *Equivalent 2-cocycles lead to isomorphic extensions.*

Dimension 4: scalar restricted 2-cocycles

Notation: Let $L = L_{\bar{0}} \oplus L_{\bar{1}} = \langle e_1, \cdots, e_n | e_{n+1}, \cdots, e_{n+m} \rangle$ be a restricted Lie superalgebra of superdimension $\mathrm{sdim}(L) = (n|m)$. A basis for (ordinary) 2-cocycles is then given by

$$\Delta_{i,j} : L \times L \longrightarrow \mathbb{K}, \qquad 1 \leq i \leq n+m, \; i \leq j \leq n+m,$$

where $\Delta_{i,j}(e_k, e_l) = \delta_{i,k}\delta_{j,l}$ and $\Delta_{i,j} = -(-1)^{|e_i||e_j|}\Delta_{j,i}$.

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Theorem

Suppose that L is a nilpotent Lie superalgebra of total dimension 3 with $\dim(L_{\bar{1}}) \geq 1$ over an algebraically closed field of characteristic $p \geq 3$. The equivalence classes of (ordinary) non trivial homogeneous 2-cocycles on L are given by

$L = \mathbf{L}^1_{0|3}$: $\Delta_{1,1}, \ \Delta_{1,2}, \ \Delta_{1,1} + \Delta_{2,3}$;

$L = \mathbf{L}^1_{1|2}$: $\Delta_{1,2}, \ \Delta_{2,3}, \ \Delta_{2,2} + \Delta_{2,3} + \Delta_{3,3}$;

$L = \mathbf{L}^2_{1|2}$: $\Delta_{2,2}, \ \Delta_{2,2} + \Delta_{3,3}$;

$L = \mathbf{L}^3_{1|2}$: $\Delta_{1,3}, \ \Delta_{2,2}$;

$L = \mathbf{L}^4_{1|2}$: $\Delta_{2,2}, \ \Delta_{2,3}, \ \Delta_{2,2} + \Delta_{2,3}$.

$L = \mathbf{L}^1_{2|1}$: $\Delta_{1,3}, \ \Delta_{1,2}, \ \Delta_{3,3}, \ \Delta_{1,2} + \Delta_{3,3}$;

$L = \mathbf{L}^2_{2|1}$: $\Delta_{1,3}$.

Dimension 4: the classification. Building the extensions.

With the list of 2-cocycles, we can extend the Lie brackets using

$$[x, y]_{\text{new}} = [x, y]_{\text{old}} + \Delta(x, y)X. \tag{4}$$

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Example. Consider $\mathbf{L}^{\mathbf{3}}_{\mathbf{1}|\mathbf{2}} = \langle e_1 | e_2, e_3; [e_1, e_2] = e_3 \rangle$. The 2-cocycles are $\Delta_{1,3}$ and $\Delta_{2,2}$. We obtain four superalgebras of dimension 4.

Name	sdim	Cocycle	Added element	Bracket
$\mathbf{L}^{\mathbf{g}}_{\mathbf{2}\|\mathbf{2}}$	$(2\|2)$	0	X even	$[e_1, e_2] = e_3$
$\mathbf{L}^{\mathbf{d}}_{\mathbf{1}\|\mathbf{3}}$	$(1\|3)$	0	X odd	$[e_1, e_2] = e_3$
$\mathbf{L}^{\mathbf{e}}_{\mathbf{1}\|\mathbf{3}}$	$(1\|3)$	$\Delta_{1,3}$	X odd	$[e_1, e_2] = e_3, \ [e_1, e_3] = X$
$\mathbf{L}^{\mathbf{h}}_{\mathbf{2}\|\mathbf{2}}$	$(2\|2)$	$\Delta_{2,2}$	X even	$[e_1, e_2] = e_3, \ [e_2, e_2] = X$

Lie superalgebras obtained by central extensions of $\mathbf{L}^{\mathbf{3}}_{\mathbf{1}|\mathbf{2}}$.

Dimension 4: the classification. Detecting isomorphisms.

It is possible that two superalgebras obtained as central extensions by non-equivalent cocycles are isomorphic. We need to detect and remove redundancies.

L	$[L, L]$	$\mathrm{sdim}(\mathfrak{z}(L))$	$\mathrm{sdim}\left(H^1_{\mathrm{CE}}(L;\mathbb{K})\right)$	$\mathrm{sdim}\left(H^2_{\mathrm{CE}}(L;\mathbb{K})\right)$	$\mathrm{sdim}\left(H^3_{\mathrm{CE}}(L;\mathbb{K})\right)$
$\mathbf{L}^{\mathbf{a}}_{\mathbf{1}\vert\mathbf{3}}$	0	$1\vert 3$	$1\vert 3$	$6\vert 3$	$7\vert 9$
$\mathbf{L}^{\mathbf{b}}_{\mathbf{1}\vert\mathbf{3}}$	$\langle X\rangle$	$0\vert 2$	$1\vert 2$	$3\vert 2$	$3\vert 4$ ($3\vert 5$ if $p=3$)
$\mathbf{L}^{\mathbf{c}}_{\mathbf{1}\vert\mathbf{3}}$	$\langle e_1\rangle$	$1\vert 1$	$0\vert 3$	$5\vert 0$	$0\vert 7$
$\mathbf{L}^{\mathbf{e}}_{\mathbf{1}\vert\mathbf{3}}$	$\langle e_3, X\rangle$	$0\vert 1$	$1\vert 1$	$2\vert 1$	$2\vert 2$ ($2\vert 4$ if $p=3$)
$\mathbf{L}^{\mathbf{f}}_{\mathbf{1}\vert\mathbf{3}}$	$\langle e_1\rangle$	$1\vert 2$	$0\vert 3$	$5\vert 0$	$0\vert 7$
$\mathbf{L}^{\mathbf{j}}_{\mathbf{1}\vert\mathbf{3}}$	$\langle X\rangle$	$1\vert 0$	$0\vert 3$	$5\vert 0$	$0\vert 7$

Invariants for Lie superalgebras of sdim $= (1|3)$.

Dimension 4: the classification. Lie superalgebras.

Theorem

The classification of 4-dimensional nilpotent Lie superalgebras over an algebraically closed field of characteristic different from 2 is given by:

$\underline{sdim(L) = (0|4)}$: $L = \langle 0|x_1, x_2, x_3, x_4\rangle$

$\mathbf{L^1_{0|4}}$: $[\cdot,\cdot] = 0$.

$\underline{sdim(L) = (1|3)}$: $L = \langle x_1|x_2, x_3, x_4\rangle$

$\mathbf{L^1_{1|3}}$ $(= \mathbf{L^a_{1|3}})$: *abelian;*

$\mathbf{L^2_{1|3}}$ $(= \mathbf{L^b_{1|3}})$: $[x_1, x_3] = x_4$;

$\mathbf{L^3_{1|3}}$ $(= \mathbf{L^c_{1|3}})$: $[x_2, x_3] = x_1$;

$\mathbf{L^4_{1|3}}$ $(= \mathbf{L^e_{1|3}})$: $[x_1, x_2] = x_3$, $[x_1, x_3] = x_4$;

$\mathbf{L^5_{1|3}}$ $(= \mathbf{L^f_{1|3}})$: $[x_3, x_3] = x_1$;

$\mathbf{L^6_{1|3}}$ $(= \mathbf{L^j_{1|3}})$: $[x_2, x_2] = x_1$, $[x_3, x_4] = x_1$.

$\underline{sdim(L) = (2|2)}$: $L = \langle x_1, x_2|x_3, x_4\rangle$

$\mathbf{L^1_{2|2}}$ $(= \mathbf{L^a_{2|2}})$: *abelian;*

$\mathbf{L^2_{2|2}}$ $(= \mathbf{L^b_{2|2}})$: $[x_3, x_4] = x_2$;

$\mathbf{L^3_{2|2}}$ $(= \mathbf{L^e_{2|2}})$: $[x_3, x_3] = x_2$, $[x_3, x_4] = x_1$;

$\mathbf{L^4_{2|2}}$ $(= \mathbf{L^f_{2|2}})$: $[x_3, x_3] = [x_4, x_4] = x_2$, $[x_3, x_4] = x_1$;

$\mathbf{L^5_{2|2}}$ $(= \mathbf{L^g_{2|2}})$: $[x_1, x_3] = x_4$;

$\mathbf{L^6_{2|2}}$ $(= \mathbf{L^h_{2|2}})$: $[x_1, x_3] = x_4$, $[x_3, x_3] = x_2$.

$\mathbf{L^7_{2|2}}$ $(= \mathbf{L^i_{2|2}})$: $[x_4, x_4] = x_1$.

$\underline{sdim(L) = (3|1)}$: $L = \langle x_1, x_2, x_3|x_4\rangle$

$\mathbf{L^1_{3|1}}$ $(= \mathbf{L^a_{3|1}})$: *abelian;*

$\mathbf{L^2_{3|1}}$ $(= \mathbf{L^b_{3|1}})$: $[x_1, x_2] = x_3$;

$\mathbf{L^3_{3|1}}$ $(= \mathbf{L^c_{3|1}})$: $[x_2, x_2] = x_3$;

$\mathbf{L^4_{3|1}}$ $(= \mathbf{L^d_{3|1}})$: $[x_1, x_2] = [x_3, x_4] = x_3$.

$\underline{sdim(L) = (4|0)}$: $L = \langle x_1, x_2, x_3, x_4|0\rangle$

$\mathbf{L^1_{4|0}}$: *abelian;*

$\mathbf{L^2_{4|0}}$: $[x_1, x_2] = x_3$;

$\mathbf{L^3_{4|0}}$: $[x_1, x_2] = x_3$, $[x_1, x_3] = x_4$.

Dimension 4: the classification. $p|2p$ maps.

Theorem (Jacobson)

Let $(e_j)_{j \in J}$ be a basis of $L_{\bar{0}}$, and let the elements $f_j \in L_{\bar{0}}$ be such that

$$(\mathrm{ad}_{e_j})^p = \mathrm{ad}_{f_j}. \tag{5}$$

Then, there exists exactly one $p|2p$-mapping $(\cdot)^{[p|2p]} : L \to L$ such that

$$e_j^{[p]} = f_j \quad \text{for all } j \in J.$$

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- The classification of 4-dimensional restricted Lie algebras has been achieved by Schneider-Usefi.

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- The classification of 4-dimensional restricted Lie algebras has been achieved by Schneider-Usefi.
- We only have to check whether these p-maps satisfy Condition (5).

Dimension 4: the classification. $p|2p$ maps.

Theorem

The p-nilpotent structures on nilpotent Lie superalgebras of total dimension 4 *with* $\dim(L_{\bar{1}}) > 0$ *are given by:*

- *sdim*$(L) = (0|4)$*: none.*
- *sdim*$(L) = (1|3)$*:* $x_1^{[p]} = 0$.
- *sdim*$(L) = (2|2)$*:*
  - $x_1^{[p]_1} = x_2^{[p]_1} = 0;$
  - $x_1^{[p]_2} = x_2,\ x_2^{[p]_2} = 0.$
- *sdim*$(L) = (3|1)$*:*
  - *Case* $L_{\bar{0}}$ *abelian:*
    - $x_1^{[p]_1} = x_2^{[p]_1} = x_3^{[p]_1} = 0;$
    - $x_1^{[p]_2} = x_2,\ x_2^{[p]_2} = x_3^{[p]_2} = 0.$
    - $x_1^{[p]_3} = x_2,\ x_2^{[p]_3} = x_3,\ x_3^{[p]_3} = 0.$
  - *Case* $L_{\bar{0}} \cong \mathbf{L}^2_{3|0} = \langle x_1, x_2, x_3; [x_1, x_2] = x_3\rangle$*:*
    - $x_1^{[p]_4} = x_2^{[p]_4} = x_3^{[p]_4} = 0;$
    - $x_1^{[p]_5} = x_3,\ x_2^{[p]_5} = x_3^{[p]_5} = 0.$

Thank you for your attention!