Central extensions of restricted Lie superalgebras and classification of p-nilpotent Lie superalgebras in dimension 4

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Motivation

Let $\mathbb{K}$ be a field of characteristic $p > 2$, algebraically closed.

- **Our goals:**
  - classification of low-dimensional *p-nilpotent restricted* Lie superalgebras over $\mathbb{K}$.
  - superization of formulas for the restricted cohomology.

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- **What do we need?** Restricted 2-cocycles of the restricted cohomology for restricted Lie superalgebras.

Restricted Lie algebras

Definition (Jacobson)

A **restricted Lie algebra** *is a Lie algebra L equipped with a map $(\cdot)^{[p]} : L \longrightarrow L$ satisfying for all $x, y \in L$ and for all $\lambda \in \mathbb{K}$:*

1. $(\lambda x)^{[p]} = \lambda^p x^{[p]}$;
2. $\left[x, y^{[p]}\right] = [[\cdots[x, \overbrace{y], y], \cdots, y}^{p \text{ terms}}]$;
3. $(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x,y)$,

Nathan Jacobson (1910-1999)

with is$_i(x,y)$ the coefficient of Z^{i-1} in $\mathrm{ad}^{p-1}_{Zx+y}(x)$. Such a map $(-)^{[p]} : L \longrightarrow L$ is called p-map.

Example: any associative algebra A with $[a,b] = ab - ba$ and $a^{[p]} = a^p$, $\forall a, b \in A$.

Restricted Lie algebras

Definition

A Lie algebra morphism $f : \left(L, [\cdot,\cdot], (\cdot)^{[p]}\right) \to \left(L', [\cdot,\cdot]', (\cdot)^{[p]'}\right)$ *is called* **restricted** *if*

$$f\left(x^{[p]}\right) = f(x)^{[p]'}, \ \forall x \in L.$$

A L-module M is called **restricted** *if*

$$x^{[p]} \cdot m = \left(\overbrace{x \cdot (x \cdots (x}^{p \text{ terms}} \cdot m) \cdots)\right), \ \forall x \in L, \ \forall m \in M.$$

Restricted Lie superalgebras

Definition (Restricted Lie superalgebra)

A **restricted Lie superalgebra** *is a Lie superalgebra* $L = L_{\bar{0}} \oplus L_{\bar{1}}$ *such that*

1. *The even part* $L_{\bar{0}}$ *is a restricted Lie algebra;*
2. *The odd part* $L_{\bar{1}}$ *is a Lie* $L_{\bar{0}}$*-module;*
3. $[x, y^{[p]}] = [[\ldots[x, \overbrace{y], y], \ldots, y}^{p \text{ terms}}], \ \forall x \in L_{\bar{1}}, \ y \in L_{\bar{0}}.$

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We can define a map $(\cdot)^{[2p]} : L_{\bar{1}} \to L_{\bar{0}}$ *by*

$$x^{[2p]} = \left(x^2\right)^{[p]}, \text{ with } x^2 = \frac{1}{2}[x, x], \ x \in L_{\bar{1}}.$$

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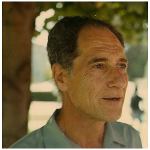
Theorem (Jacobson)

Let $(e_j)_{j \in J}$ *be a basis of* $L_{\bar{0}}$*, and let the elements* $f_j \in L_{\bar{0}}$ *be such that* $(\mathrm{ad}_{e_j})^p = \mathrm{ad}_{f_j}$*. Then, there exists exactly one* $p|2p$*-mapping* $(\cdot)^{[p|2p]} : L \to L$ *such that*

$$e_j^{[p]} = f_j \quad \textit{for all } j \in J.$$

A (very) brief history of restricted cohomology

- 1955 (Hochschild): $H_*^n(L, M) := \mathrm{Ext}^n_{U_p(L)}(\mathbb{F}, M)$.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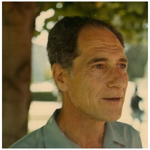

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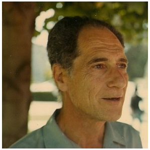

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- 2020 (Yuan-Chen-Cao): attempt to generalize to the superalgebras case.

Restricted cohomology for restricted Lie superalgebras

Let $L = L_{\bar{0}} \oplus L_{\bar{1}}$ be a restricted Lie superalgebra and let M be a L-supermodule.

We set $C^0_*(L, M) = M$ and $C^1_*(L, M) = \mathrm{Hom}(L, M)$.

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Definition (Restricted 2-cochains)

Let $\varphi \in C^2_{CE}(L, M)$ *(ordinary Chevalley-Eilenberg 2-cochain) and* $\omega : L \longrightarrow M$*.*
Then ω *is* φ-**compatible** *if*

1. $\omega(\lambda x) = \lambda^p \omega(x), \ \forall \lambda \in \mathbb{F}, \ \forall x \in L_{\bar{0}};$

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2. $\omega(x + y) = \omega(x) + \omega(y) +$
$$\sum_{\substack{x_i = x \text{ or } y \\ x_1 = x, \ x_2 = y}} \frac{1}{\pi(x)} \sum_{k=0}^{p-2} (-1)^k x_p \cdots x_{p-k+1} \varphi\big([[\cdots[x_1, x_2], x_3] \cdots, x_{p-k-1}], x_{p-k}\big),$$

with $x, y \in L_{\bar{0}}$, $\pi(x)$ the number of factors x_i equal to x.

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with $x, y \in L_{\bar{0}}$, $\pi(x)$ the number of factors x_i equal to x.

$$C^2_*(L, M) := \Big\{(\varphi, \omega), \ \varphi \in C^2_{CE}(L, M), \ \omega \text{ is } \varphi\text{-compatible}\Big\}$$

$\rightsquigarrow$ We have a similar (although more complicated) definition for $C^3_*(L, M)$.

Restricted cohomology for restricted Lie superalgebras

- A **restricted 2-cocycle** is an element $(\alpha, \beta) \in C^2_*(L, M)$ such that

  1. $(-1)^{|x||z|}\alpha(x, [y, z]) + (-1)^{|y||x|}\alpha(y, [z, x]) + (-1)^{|z||y|}\alpha(z, [x, y]) = 0,$
  $\forall x, y, z \in L;$

  2. $$\alpha\left(x, y^{[p]}\right) - \sum_{i+j=p-1} (-1)^i y^i \alpha\Big([x, \underbrace{y, \cdots, y}_{j \text{ terms}}], y\Big) + (-1)^{|x||\alpha|} x\beta(y) = 0,$$
  $\forall x \in L,\ y \in L_{\bar{0}}.$

  The space of restricted 2-cocycles is denoted by $Z^2_*(L, M)$.

Restricted cohomology for restricted Lie superalgebras

- A **restricted 2-cocycle** is an element $(\alpha, \beta) \in C_*^2(L, M)$ such that

  1. $(-1)^{|x||z|}\alpha(x, [y, z]) + (-1)^{|y||x|}\alpha(y, [z, x]) + (-1)^{|z||y|}\alpha(z, [x, y]) = 0,$ $\forall x, y, z \in L;$

  2. $\alpha\left(x, y^{[p]}\right) - \sum_{i+j=p-1} (-1)^i y^i \alpha\Big([x, \underbrace{y, \cdots, y}_{j \text{ terms}}], y\Big) + (-1)^{|x||\alpha|} x\beta(y) = 0,$ $\forall x \in L,\ \ y \in L_{\bar{0}}.$

  The space of restricted 2-cocycles is denoted by $Z_*^2(L, M)$.

- A **restricted 2-coboundary** is an element $(\alpha, \beta) \in C_*^2(L, M)$ such that $\exists \varphi \in \mathrm{Hom}(L, M)$,

  1. $\alpha(x, y) = \varphi\big([x, y]\big) - x\varphi(y) + y\varphi(x),\ \forall x, y \in L;$

  2. $\beta(x) = \varphi\left(x^{[p]}\right) - x^{p-1}\varphi(x), \forall x \in L_{\bar{0}}.$

  The space of restricted 2-coboundaries is denoted by $B_*^2(L, M)$.

Restricted cohomology for restricted Lie superalgebras

The previous formulae define maps

$$0 \longrightarrow C^0_*(L,M) \xrightarrow{d^0_*} C^1_*(L,M) \xrightarrow{d^1_*} C^2_*(L,M) \xrightarrow{d^2_*} C^3_*(L,M),$$

with $d^0_* = d^0_{CE}$.

Theorem

We have $d^2_* \circ d^1_* = 0$. Therefore, the quotient space

$$H^2_*(L;M) = Z^2_*(L;M)/B^2_*(L;M)$$

is well defined.

Restricted cohomology for restricted Lie superalgebras

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Let L be a restricted Lie superalgebra and M a restricted L-module. We define a subspace $C^2_*(L; M)^+ \subset C^2_*(L; M)$ by

$$C^2_*(L; M)^+ := \left\{(\alpha, \beta) \in C^2_*(L; M),\ \mathrm{Im}(\beta) \subseteq M_{\bar{0}}\right\}.$$

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Lemma

(i) *We have an inclusion* $B^2_*(L;M)_{\bar{0}} \subset C^2_*(L;M)^+$.

(ii) *The space* $C^2_*(L;M)^+$ *is* $\mathbb{Z}_2$*-graded and the degree of an homogeneous element* $(\alpha,\beta) \in C^2_*(L;M)^+$ *is given by* $|(\alpha,\beta)| = |\alpha|$.

This Lemma allows us to consider the space $Z^2_*(L;M)^+ := \ker\big(d^2_{*|C^2_*(L;M)^+}\big)$. Thus we can define

$$H^2_*(L;M)^+ := Z^2_*(L;M)^+/B^2_*(L;M)_{\bar{0}}.$$

The space $H^2_*(L;M)^+$ is $\mathbb{Z}_2$-graded.

Central extensions of restricted Lie superalgebras

Let $\left(L, [\cdot,\cdot], (\cdot)^{[p]}\right)$ be a restricted Lie superalgebra, and M be a strongly abelian restricted Lie superalgebra $\left(\textit{i.e}, [m,n]=0\ \forall m,n \in M, \text{ and } m^{[p]}=0\ \forall m \in M_{\bar{0}}\right)$.

A **restricted extension** of L by M is a short exact sequence of restricted Lie superalgebras

$$0 \longrightarrow M \xrightarrow{\iota} E \xrightarrow{\pi} L \longrightarrow 0.$$

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A **restricted extension** of L by M is a short exact sequence of restricted Lie superalgebras

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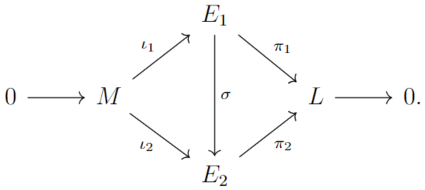
In the case where $\iota(M) \subset \mathfrak{z}(E) := \{a \in E,\ [a,b] = 0\ \forall b \in E\}$, M is a trivial L-module. These extensions are called **restricted central extensions**.

Two restricted central extensions of L by M are called **equivalent** if there is a restricted Lie superalgebras morphism $\sigma : E_1 \to E_2$ such that the following diagram commutes:



Central extensions of restricted Lie superalgebras

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Theorem

Let L be a restricted Lie superalgebra and M a strongly abelian restricted Lie superalgebra. Then, the equivalence classes of restricted central extensions of L by M are classified by $H^2_*(L; M)^+_{\bar{0}}$.

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Theorem

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Structure maps on E. Let $(\varphi, \omega) \in Z^2_*(L; M)^+_{\bar{0}}$. The bracket and the p- map on E are given by

$$[x+m, y+n]_E := [x, y] + \varphi(x, y), \qquad \forall x, y \in L, \ \forall m, n \in M; \tag{1}$$

$$(x+m)^{[p]_E} := (x)^{[p]} + \omega(x), \qquad \forall x \in L_{\bar{0}}, \ \forall m \in M_{\bar{0}}. \tag{2}$$

A brief history of classification of restricted Lie algebras

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Proposition

Let L be a p-nilpotent restricted Lie superalgebra of dimension n. Then, L is isomorphic to a central extension by a restricted 2-cocycle of a p-nilpotent restricted Lie superalgebra of dimension $n-1$.

Dimension 3

- sdim$(L) = (1|2)$: $L = \langle e_1 | e_2, e_3 \rangle$.

  1. $\mathbf{L}^1_{1|2} = \langle e_1 | e_2, e_3 \rangle$ (abelian):
     1. $e_1^{[p]} = 0$;
  2. $\mathbf{L}^2_{1|2} = \langle e_1 | e_2, e_3; [e_2, e_3] = e_1 \rangle$:
     1. $e_1^{[p]} = 0$;
  3. $\mathbf{L}^3_{1|2} = \langle e_1 | e_2, e_3; [e_1, e_2] = e_3 \rangle$:
     1. $e_1^{[p]} = 0$.
  4. $\mathbf{L}^4_{1|2} = \langle e_1 | e_2, e_3; [e_3, e_3] = e_1 \rangle$:
     1. $e_1^{[p]} = 0$;

- sdim$(L) = (2|1)$: $L = \langle e_1, e_2 | e_3 \rangle$.

  1. $\mathbf{L}^1_{2|1} = \langle e_1, e_2 | e_3 \rangle$ (abelian):
     1. $e_1^{[p]} = e_2^{[p]} = 0$;
     2. $e_1^{[p]} = e_2, \; e_2^{[p]} = 0$.
  2. $\mathbf{L}^2_{2|1} = \langle e_1, e_2 | e_3; [e_3, e_3] = e_2 \rangle$:
     1. $e_1^{[p]} = e_2^{[p]} = 0$;
     2. $e_1^{[p]} = e_2, \; e_2^{[p]} = 0$.

- sdim$(L) = (3|0)$: $L = \langle e_1, e_2, e_3 \rangle$, (see Schneider-Usefi).

  1. $\mathbf{L}^1_{3|0} = \langle e_1, e_2, e_3 \rangle$ (abelian):
     1. $e_1^{[p]} = e_2^{[p]} = e_3^{[p]} = 0$;
     2. $e_1^{[p]} = e_2, \; e_2^{[p]} = e_3^{[p]} = 0$;
     3. $e_1^{[p]} = e_2, \; e_2^{[p]} = e_3, \; e_3^{[p]} = 0$.
  2. $\mathbf{L}^2_{3|0} = \langle e_1, e_2, e_3; [e_1, e_2] = e_3 \rangle$
     1. $e_1^{[p]} = e_2^{[p]} = e_3^{[p]} = 0$;
     2. $e_1^{[p]} = e_3, \; e_2^{[p]} = e_3^{[p]} = 0$.

The classification method

1. For each 3-dimensional Lie superalgebra of the previous list, we compute the equivalence classes of non-trivial *ordinary* 2-cocycles under the action by automorphisms given by

$$(A\varphi)(x,y) = \varphi\big(A(x), A(y)\big), \ \forall x, y \in L \tag{3}$$

2. We build the corresponding central extensions.

3. Some of the superalgebras obtained are isomorphic. We detect and remove redundancies.

4. Using Jacobson's Theorem, we check whether the p-maps on the even part are compatible with the odd part.

Dimension 4: the classification. Lie superalgebras.

Theorem

The classification of 4-dimensional nilpotent Lie superalgebras over an algebraically closed field of characteristic different from 2 is given by:

$\underline{sdim(L) = (0|4)}$: $L = \langle 0|x_1, x_2, x_3, x_4\rangle$

$\mathbf{L^1_{0|4}}$: $[\cdot,\cdot] = 0$.

$\underline{sdim(L) = (1|3)}$: $L = \langle x_1|x_2, x_3, x_4\rangle$

$\mathbf{L^1_{1|3}}$: *abelian;*

$\mathbf{L^2_{1|3}}$: $[x_1, x_3] = x_4$;

$\mathbf{L^3_{1|3}}$: $[x_2, x_3] = x_1$;

$\mathbf{L^4_{1|3}}$: $[x_1, x_2] = x_3,\ [x_1, x_3] = x_4$;

$\mathbf{L^5_{1|3}}$: $[x_3, x_3] = x_1$;

$\mathbf{L^6_{1|3}}$: $[x_2, x_2] = x_1,\ [x_3, x_4] = x_1$.

$\underline{sdim(L) = (2|2)}$: $L = \langle x_1, x_2|x_3, x_4\rangle$

$\mathbf{L^1_{2|2}}$: *abelian;*

$\mathbf{L^2_{2|2}}$: $[x_3, x_4] = x_2$;

$\mathbf{L^3_{2|2}}$: $[x_3, x_3] = x_2,\ [x_3, x_4] = x_1$;

$\mathbf{L^4_{2|2}}$: $[x_3, x_3] = [x_4, x_4] = x_2,\ [x_3, x_4] = x_1$;

$\mathbf{L^5_{2|2}}$: $[x_1, x_3] = x_4$;

$\mathbf{L^6_{2|2}}$: $[x_1, x_3] = x_4,\ [x_3, x_3] = x_2$.

$\mathbf{L^7_{2|2}}$: $[x_4, x_4] = x_1$.

$\underline{sdim(L) = (3|1)}$: $L = \langle x_1, x_2, x_3|x_4\rangle$

$\mathbf{L^1_{3|1}}$: *abelian;*

$\mathbf{L^2_{3|1}}$: $[x_1, x_2] = x_3$;

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$\underline{sdim(L) = (4|0)}$: $L = \langle x_1, x_2, x_3, x_4|0\rangle$

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$\mathbf{L^3_{4|0}}$: $[x_1, x_2] = x_3,\ [x_1, x_3] = x_4$.

Dimension 4: the classification. $p|2p$ maps.

Theorem

The p-nilpotent structures on nilpotent Lie superalgebras of total dimension 4 *with* $\dim(L_{\bar{1}}) > 0$ *are given by:*

- $sdim(L) = (0|4)$: *none.*
- $sdim(L) = (1|3)$: $x_1^{[p]} = 0$.
- $sdim(L) = (2|2)$:
  - $x_1^{[p]_1} = x_2^{[p]_1} = 0$;
  - $x_1^{[p]_2} = x_2,\ x_2^{[p]_2} = 0$.
- $sdim(L) = (3|1)$:
  - *Case* $L_{\bar{0}}$ *abelian:*
    - $x_1^{[p]_1} = x_2^{[p]_1} = x_3^{[p]_1} = 0$;
    - $x_1^{[p]_2} = x_2,\ x_2^{[p]_2} = x_3^{[p]_2} = 0$.
    - $x_1^{[p]_3} = x_2,\ x_2^{[p]_3} = x_3,\ x_3^{[p]_3} = 0$.
  - *Case* $L_{\bar{0}} \cong \mathbf{L}^{\mathbf{2}}_{\mathbf{3|0}} = \langle x_1, x_2, x_3; [x_1, x_2] = x_3 \rangle$:
    - $x_1^{[p]_4} = x_2^{[p]_4} = x_3^{[p]_4} = 0$;
    - $x_1^{[p]_5} = x_3,\ x_2^{[p]_5} = x_3^{[p]_5} = 0$.

Thank you for your attention!