Central extensions of restricted Lie superalgebras and classification of p-nilpotent Lie superalgebras in dimension 4

Quentin Ehret

Mulhouse, Mars 2024

Joint work with Sofiane Bouarroudj

جامعة نيويورك أبوظبي
NYU | ABU DHABI

Outline

Let $\mathbb{K}$ be a field of characteristic $p > 2$, algebraically closed.

- **Our goals:**
  - classification of low-dimensional *p-nilpotent restricted* Lie superalgebras over $\mathbb{K}$.
  - superization of formulas for the restricted cohomology.

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- **What do we need?** Restricted 2-cocycles of the restricted cohomology for restricted Lie superalgebras.

Restricted Lie algebras

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Then, if $m = p$, we obtain

$$\mathrm{ad}_x^p(y) = x^p y - y x^p = \mathrm{ad}_{x^p}(y).$$

Restricted Lie algebras

Definition (Jacobson)

A **restricted Lie algebra** *is a Lie algebra L equipped with a map $(\cdot)^{[p]} : L \longrightarrow L$ satisfying for all $x, y \in L$ and for all $\lambda \in \mathbb{K}$:*

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3. $(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x, y)$,

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Example: any associative algebra A with $[a, b] = ab - ba$ and $a^{[p]} = a^p, \ \forall a, b \in A$.

Restricted Lie algebras

Very useful :

$$\sum_{i=1}^{p-1} s_i(x,y) = \sum_{\substack{x_i = x \text{ or } y \\ x_p = x,\ x_{p-1} = y}} \frac{1}{\sharp\{x\}} [x_1, [x_2, [\ldots, [x_{p-1}, x_p]\ldots],$$

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Definition

A Lie algebra morphism $f : \left(L, [\cdot,\cdot], (\cdot)^{[p]}\right) \to \left(L', [\cdot,\cdot]', (\cdot)^{[p]'}\right)$ *is called* **restricted** *if*

$$f\left(x^{[p]}\right) = f(x)^{[p]'},\ \forall x \in L.$$

A L*-module* M *is called* **restricted** *if*

$$x^{[p]} \cdot m = \Big(\overbrace{x \cdot (x \cdots (x}^{p \text{ terms}} \cdot m) \cdots)\Big),\ \forall x \in L,\ \forall m \in M.$$

Lie superalgebras

Definition

A Lie superalgebra $L = L_{\bar{0}} \oplus L_{\bar{1}}$ *is a* $\mathbb{Z}/2\mathbb{Z}$*-graded vector space equipped with a bilinear map* $[\cdot,\cdot] : L \times L \to L$ *satisfying for* $x, y, z \in L$ *:*

1. $|[x,y]| = |x| + |y|$;
2. $[x,y] = -(-1)^{|x||y|}[y,x]$;
3. $(-1)^{|x||z|}[x,[y,z]] + (-1)^{|x||y|}[y,[z,x]] + (-1)^{|y||z|}[z,[x,y]] = 0.$

If $p = 3$, the identity $[x,[x,x]] = 0,\ x \in L_{\bar{1}}$ has to be added as an axiom as well.

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Let $f : V \to W$ be a map between $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces. Then:

- the map f is called **even** if $f(V_{\bar{i}}) \subset W_{\bar{i}}$;
- the map f is called **odd** if $f(V_{\bar{i}}) \subset W_{\overline{i+1}}$;

Restricted Lie superalgebras

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A **restricted Lie superalgebra** *is a Lie superalgebra* $L = L_{\bar{0}} \oplus L_{\bar{1}}$ *such that*

1. *The even part* $L_{\bar{0}}$ *is a restricted Lie algebra;*
2. *The odd part* $L_{\bar{1}}$ *is a Lie* $L_{\bar{0}}$*-module;*
3. $$[x, y^{[p]}] = [[\ldots[x, \overbrace{y], y], \ldots, y}^{p \text{ terms}}], \ \forall x \in L_{\bar{1}}, \ y \in L_{\bar{0}}.$$

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We can define a map $(\cdot)^{[2p]} : L_{\bar{1}} \to L_{\bar{0}}$ *by*

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Theorem (Jacobson)

Let $(e_j)_{j \in J}$ *be a basis of* $L_{\bar{0}}$*, and let the elements* $f_j \in L_{\bar{0}}$ *be such that* $(\mathrm{ad}_{e_j})^p = \mathrm{ad}_{f_j}$*. Then, there exists exactly one* $p|2p$*-mapping* $(\cdot)^{[p|2p]} : L \to L$ *such that*

$$e_j^{[p]} = f_j \quad \textit{for all } j \in J.$$

Restricted p-nilpotent Lie superalgebras

Let L be a Lie superalgebra. We define a *descending central sequence* by

$$C^0(L) = L, \quad \text{and} \quad C^{k+1}(L) = \left[C^k(L), L\right].$$

The Lie superalgebra L is called *nilpotent* if there exists $k \geq 0$ such that $C^k(L) = 0$.

Suppose that L is restricted. Then L is called *p-nilpotent* if there exists $n \geq 0$ such that $x^{[p]^n} = 0 \ \forall x \in L_{\bar{0}}$. Any p-nilpotent restricted Lie superalgebra is nilpotent.

Chevalley-Eilenberg cohomology for Lie superalgebras

Let $L = L_{\bar{0}} \oplus L_{\bar{1}}$ be a restricted Lie superalgebra and let $M = M_{\bar{0}} \oplus M_{\bar{1}}$ be a restricted module.
For $n = 0$: $C^0_{\mathrm{CE}}(L, M) := M$.
For $n > 0$: $C^n_{\mathrm{CE}}(L, M)$ is the space of n-linear super anti-symmetric maps with values in M.

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$$d^0_{\mathrm{CE}}(m)(x) = (-1)^{|m||x|} x \cdot m \quad \forall m \in M \text{ and } \forall x \in L;$$

$$d^n_{\mathrm{CE}}(\varphi)(x_1, \dots, x_n)$$

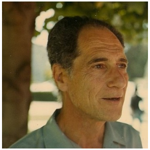
$$= \sum_{i<j} (-1)^{|x_j|(|x_{i+1}|+\cdots+|x_{j-1}|)+j} \varphi(x_1, \dots, x_{i-1}, [x_i, x_j], x_{i+1} \dots, \widetilde{x}_j, \dots, x_n)$$
$$+ \sum_{j} (-1)^{|x_j|(|\varphi|+|x_1|+\cdots+|x_{j-1}|)+j} x_j \cdot \varphi(x_1, \dots, \widetilde{x}_j, \dots, x_n)$$

for any $\varphi \in C^{n-1}_{\mathrm{CE}}(L; M)$ with $n > 0$, and $x_1, \dots, x_n \in L$.

The spaces $C^n_{\mathrm{CE}}(L; M)$ are $\mathbb{Z}_2$-graded.

A (very) brief history of restricted cohomology

- 1955 (Hochschild): $H^n_*(L,M) := \mathrm{Ext}^n_{U(L)}(\mathbb{F}, M)$.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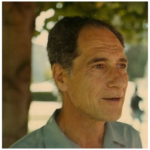

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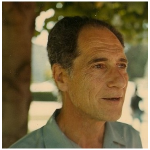
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- 2020 (Yuan-Chen-Cao): attempt to generalize to the superalgebras case.

Restricted cohomology for restricted Lie superalgebras

Let $L = L_{\bar{0}} \oplus L_{\bar{1}}$ be a restricted Lie superalgebra and let M be a L-supermodule.

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Definition (Restricted 2-cochains)

*Let $\varphi \in C^2_{CE}(L, M)$ (ordinary Chevalley-Eilenberg 2-cochain) and $\omega : L \longrightarrow M$. Then ω is φ-**compatible** if*

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$$\omega(x) + \omega(y) + \sum_{\substack{x_i = x \text{ or } y \\ x_1 = x, \ x_2 = y}} \frac{1}{\pi(x)} \sum_{k=0}^{p-2} (-1)^k x_p \dots x_{p-k+1} \varphi\big([[\dots[x_1, x_2], x_3]\dots, x_{p-k-1}], x_{p-k}\big),$$

with $x, y \in L$, $\pi(x)$ the number of factors x_i equal to x.

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with $x, y \in L$, $\pi(x)$ the number of factors x_i equal to x.

$$C^2_*(L, M) = \big\{(\varphi, \omega),\ \varphi \in C^2_{CE}(L, M),\ \omega \text{ is } \varphi\text{-compatible}\big\}$$

$\rightsquigarrow$ We have a similar (although more complicated) definition for $C^3_*(L, M)$.

Restricted cohomology for restricted Lie superalgebras

- A **restricted 2-cocycle** is an element $(\alpha, \beta) \in C_*^2(L, M)$ such that

  1. the map α is an ordinary Chevalley-Eilenberg 2-cocycle;
  2. $\alpha\left(x, y^{[p]}\right) - \sum_{i+j=p-1} (-1)^i y^i \alpha\Big([x, \underbrace{y, \cdots, y}_{j \text{ terms}}], y\Big) + (-1)^{|x||\alpha|} x\beta(y) = 0,$

  $\forall x \in L,\ y \in L_{\bar{0}}.$

  The space of restricted 2-cocycle is denoted by $Z_*^2(L, M)$.

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  $\forall x \in L,\ y \in L_{\bar{0}}.$

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- A **restricted 2-coboundary** is an element $(\alpha, \beta) \in C^2_*(L, M)$ such that $\exists \varphi \in \mathrm{Hom}(L, M)$,

  1. $\alpha(x, y) = \varphi\big([x, y]\big) - x\varphi(y) + y\varphi(x),\ \forall x, y \in L;$
  2. $\beta(x) = \varphi\left(x^{[p]}\right) - x^{p-1}\varphi(x), \forall x \in L_{\bar{0}}.$

  The space of restricted 2-coboundaries is denoted by $B^2_*(L, M)$.

Restricted cohomology for restricted Lie superalgebras

The previous formulae define maps

$$0 \longrightarrow C^0_*(L,M) \xrightarrow{d^0_*} C^1_*(L,M) \xrightarrow{d^1_*} C^2_*(L,M) \xrightarrow{d^2_*} C^3_*(L,M),$$

with $d^0_* = d^0_{CE}$.

Theorem

We have $d^2_* \circ d^1_* = 0$. Therefore, the quotient space

$$H^2_*(L;M) = Z^2_*(L;M)/B^2_*(L;M)$$

is well defined.

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Let L be a restricted Lie superalgebra and M a restricted L-module. We define a subspace $C^2_*(L;M)^+ \subset C^2_*(L;M)$ by

$$C^2_*(L;M)^+ := \Big\{(\alpha,\beta) \in C^2_*(L;M),\ \mathrm{Im}(\beta) \subseteq M_{\bar{0}}\Big\}.$$

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Lemma

(i) *We have an inclusion* $B^2_*(L;M)_{\bar{0}} \subset C^2_*(L;M)^+$.

(ii) *The space* $C^2_*(L;M)^+$ *is* $\mathbb{Z}_2$*-graded and the degree of an homogeneous element* $(\alpha,\beta) \in C^2_*(L;M)^+$ *is given by* $|(\alpha,\beta)| = |\alpha|$.

This Lemma allows us to consider the space $Z^2_*(L;M)^+ := \ker\big(d^2_{*|C^2_*(L;M)^+}\big)$. Thus we can define

$$H^2_*(L;M)^+ := Z^2_*(L;M)^+ / B^2_*(L;M)_{\bar{0}}.$$

The space $H^2_*(L;M)^+$ is $\mathbb{Z}_2$-graded.

Central extensions of restricted Lie superalgebras

Let $\left(L, [\cdot,\cdot], (\cdot)^{[p]}\right)$ be a restricted Lie superalgebra, and M be a strongly abelian restricted Lie superalgebra $\left(i.e,\ [m,n] = 0\ \forall m, n \in M,\ \text{and } m^{[p]} = 0\ \forall m \in M_{\bar{0}}\right)$.

A **restricted extension** of L by M is a short exact sequence of restricted Lie superalgebras

$$0 \longrightarrow M \xrightarrow{\iota} E \xrightarrow{\pi} L \longrightarrow 0.$$

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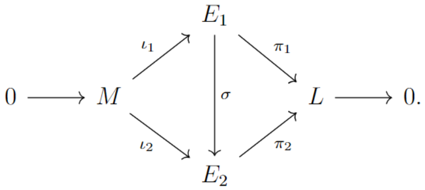
In the case where $\iota(M) \subset \mathfrak{z}(E) := \{a \in E,\ [a,b] = 0\ \forall b \in E\}$, M is a trivial L-module. These extensions are called **restricted central extensions**.

Two restricted central extensions of L by M are called **equivalent** if there is a restricted Lie superalgebras morphism $\sigma : E_1 \to E_2$ such that the following diagram commutes:



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Structure maps on E. Let $(\varphi, \omega) \in Z^2_*(L; \mathbb{K})^+_{\bar{0}}$. The bracket and the p- map on E are given by

$$[x+m, y+n]_E := [x, y] + \varphi(x, y), \ \forall x, y \in L, \ \forall m, n \in M; \quad (1)$$

$$(x+m)^{[p]_E} := (x)^{[p]} + \omega(x), \ \forall x \in L_{\bar{0}}, \ \forall m \in M_{\bar{0}}. \quad (2)$$

A brief history of classification of restricted Lie algebras

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- 2023 (Maletesta and Siciliano): Classification of p-nilpotent restricted Lie algebras of dimension 5, $p > 3$, *using another method* (J. Algebra).


Hamid Usefi


Salvatore Siciliano

Dimension 3

- sdim$(L) = (1|2)$: $L = \langle e_1 | e_2, e_3 \rangle$.
  1. $\mathbf{L}^1_{1|2} = \langle e_1 | e_2, e_3 \rangle$ (abelian):
     1. $e_1^{[p]} = 0$;
  2. $\mathbf{L}^2_{1|2} = \langle e_1 | e_2, e_3; [e_2, e_3] = e_1 \rangle$:
     1. $e_1^{[p]} = 0$;
  3. $\mathbf{L}^3_{1|2} = \langle e_1 | e_2, e_3; [e_1, e_2] = e_3 \rangle$:
     1. $e_1^{[p]} = 0$.
  4. $\mathbf{L}^4_{1|2} = \langle e_1 | e_2, e_3; [e_3, e_3] = e_1 \rangle$:
     1. $e_1^{[p]} = 0$;

- sdim$(L) = (2|1)$: $L = \langle e_1, e_2 | e_3 \rangle$.
  1. $\mathbf{L}^1_{2|1} = \langle e_1, e_2 | e_3 \rangle$ (abelian):
     1. $e_1^{[p]} = e_2^{[p]} = 0$;
     2. $e_1^{[p]} = e_2, \ e_2^{[p]} = 0$.
  2. $\mathbf{L}^2_{2|1} = \langle e_1, e_2 | e_3; [e_3, e_3] = e_2 \rangle$:
     1. $e_1^{[p]} = e_2^{[p]} = 0$;
     2. $e_1^{[p]} = e_2, \ e_2^{[p]} = 0$.

- sdim$(L) = (3|0)$: $L = \langle e_1, e_2, e_3 \rangle$, (see Schneider-Usefi).
  1. $\mathbf{L}^1_{3|0} = \langle e_1, e_2, e_3 \rangle$ (abelian):
     1. $e_1^{[p]} = e_2^{[p]} = e_3^{[p]} = 0$;
     2. $e_1^{[p]} = e_2, \ e_2^{[p]} = e_3^{[p]} = 0$;
     3. $e_1^{[p]} = e_2, \ e_2^{[p]} = e_3, \ e_3^{[p]} = 0$.
  2. $\mathbf{L}^2_{3|0} = \langle e_1, e_2, e_3; [e_1, e_2] = e_3 \rangle$
     1. $e_1^{[p]} = e_2^{[p]} = e_3^{[p]} = 0$;
     2. $e_1^{[p]} = e_3, \ e_2^{[p]} = e_3^{[p]} = 0$.

The classification method

1. For each 3-dimensional Lie superalgebra of the previous list, we compute the equivalence classes of non-trivial *ordinary* 2-cocycles under the action by automorphisms given by

$$(A\varphi)(x, y) = \varphi\big(A(x), A(y)\big), \ \forall x, y \in L \tag{3}$$

2. We build the corresponding central extensions.

3. Some of the superalgebras obtained are isomorphic. We detect and remove redundancies.

4. Using Jacobson's Theorem, we check whether the p-maps on the even part are compatible with the odd part.

Dimension 4: scalar restricted 2-cocycles

Let L be a restricted Lie superalgebra and M be a restricted L-module. Let $(\varphi, \omega) \in Z^2_*(L; M)$ and A be a restricted automorphism of L. An action is given by $A \cdot (\varphi, \omega) := (A\varphi, A\omega)$, with

$$\begin{cases} (A\varphi)(x,y) & = \varphi\big(A(x), A(y)\big), \ \forall x, y \in L \\ (A\omega)(x) & = \omega\big(A(x)\big), \ \forall x \in L_{\bar{0}}. \end{cases} \tag{4}$$

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Proposition

- *Let L be a p-nilpotent restricted Lie superalgebra of dimension n. Then, L is isomorphic to a central extension by a restricted 2-cocycle of a p-nilpotent restricted Lie superalgebra of dimension $n-1$.*
- *Equivalent 2-cocycles lead to isomorphic extensions.*

Dimension 4: scalar restricted 2-cocycles

Notation: Let $L = L_{\bar{0}} \oplus L_{\bar{1}} = \langle e_1, \cdots, e_n | e_{n+1}, \cdots, e_{n+m} \rangle$ be a restricted Lie superalgebra of superdimension $\mathrm{sdim}(L) = (n|m)$. A basis for (ordinary) 2-cocycles is then given by

$$\Delta_{i,j} : L \times L \longrightarrow \mathbb{K}, \qquad 1 \leq i \leq n+m, \; i \leq j \leq n+m,$$

where $\Delta_{i,j}(e_k, e_l) = \delta_{i,k}\delta_{j,l}$ and $\Delta_{i,j} = -(-1)^{|e_i||e_j|}\Delta_{j,i}$.

Dimension 4: scalar restricted 2-cocycles

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Theorem

Suppose that L is a nilpotent Lie superalgebra of total dimension 3 with $\dim(L_{\bar{1}}) \geq 1$ over an algebraically closed field of characteristic $p \geq 3$. The equivalence classes of (ordinary) non trivial homogeneous 2-cocycles on L are given by

$L = \mathbf{L}^1_{0|3}$: $\Delta_{1,1}, \; \Delta_{1,2}, \; \Delta_{1,1} + \Delta_{2,3}$;

$L = \mathbf{L}^1_{1|2}$: $\Delta_{1,2}, \; \Delta_{2,3}, \; \Delta_{2,2} + \Delta_{2,3} + \Delta_{3,3}$;

$L = \mathbf{L}^2_{1|2}$: $\Delta_{2,2}, \; \Delta_{2,2} + \Delta_{3,3}$;

$L = \mathbf{L}^3_{1|2}$: $\Delta_{1,3}, \; \Delta_{2,2}$;

$L = \mathbf{L}^4_{1|2}$: $\Delta_{2,2}, \; \Delta_{2,3}, \; \Delta_{2,2} + \Delta_{2,3}$.

$L = \mathbf{L}^1_{2|1}$: $\Delta_{1,3}, \; \Delta_{1,2}, \; \Delta_{3,3}, \; \Delta_{1,2} + \Delta_{3,3}$;

$L = \mathbf{L}^2_{2|1}$: $\Delta_{1,3}$.

Dimension 4: the classification. Building the extensions.

With the list of 2-cocycles, we can extend the Lie brackets using

$$[x, y]_{\text{new}} = [x, y]_{\text{old}} + \Delta(x, y)X. \tag{5}$$

Dimension 4: the classification. Building the extensions.

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$$[x, y]_{\text{new}} = [x, y]_{\text{old}} + \Delta(x, y)X. \quad (5)$$

Example. Consider $\mathbf{L}^{\mathbf{3}}_{\mathbf{1|2}} = \langle e_1 | e_2, e_3; [e_1, e_2] = e_3 \rangle$. The 2-cocycles are $\Delta_{1,3}$ and $\Delta_{2,2}$. We obtain four superalgebras of dimension 4.

Name	sdim	Cocycle	Added element	Bracket
$\mathbf{L}^{\mathbf{g}}_{\mathbf{2\|2}}$	$(2\|2)$	0	X even	$[e_1, e_2] = e_3$
$\mathbf{L}^{\mathbf{d}}_{\mathbf{1\|3}}$	$(1\|3)$	0	X odd	$[e_1, e_2] = e_3$
$\mathbf{L}^{\mathbf{e}}_{\mathbf{1\|3}}$	$(1\|3)$	$\Delta_{1,3}$	X odd	$[e_1, e_2] = e_3, \ [e_1, e_3] = X$
$\mathbf{L}^{\mathbf{h}}_{\mathbf{2\|2}}$	$(2\|2)$	$\Delta_{2,2}$	X even	$[e_1, e_2] = e_3, \ [e_2, e_2] = X$

Lie superalgebras obtained by central extensions of $\mathbf{L}^{\mathbf{3}}_{\mathbf{1|2}}$.

Dimension 4: the classification. Detecting isomorphisms.

It is possible that two superalgebras obtained as central extensions by non-equivalent cocycles are isomorphic. We need to detect and remove redundancies.

L	$[L, L]$	$\mathrm{sdim}(\mathfrak{z}(L))$	$\mathrm{sdim}\left(H^1_{\mathrm{CE}}(L;\mathbb{K})\right)$	$\mathrm{sdim}\left(H^2_{\mathrm{CE}}(L;\mathbb{K})\right)$	$\mathrm{sdim}\left(H^3_{\mathrm{CE}}(L;\mathbb{K})\right)$
$\mathbf{L}^{\mathbf{a}}_{\mathbf{1\|3}}$	0	$1\|3$	$1\|3$	$6\|3$	$7\|9$
$\mathbf{L}^{\mathbf{b}}_{\mathbf{1\|3}}$	$\langle X\rangle$	$0\|2$	$1\|2$	$3\|2$	$3\|4$ ($3\|5$ if $p=3$)
$\mathbf{L}^{\mathbf{c}}_{\mathbf{1\|3}}$	$\langle e_1\rangle$	$1\|1$	$0\|3$	$5\|0$	$0\|7$
$\mathbf{L}^{\mathbf{e}}_{\mathbf{1\|3}}$	$\langle e_3, X\rangle$	$0\|1$	$1\|1$	$2\|1$	$2\|2$ ($2\|4$ if $p=3$)
$\mathbf{L}^{\mathbf{f}}_{\mathbf{1\|3}}$	$\langle e_1\rangle$	$1\|2$	$0\|3$	$5\|0$	$0\|7$
$\mathbf{L}^{\mathbf{j}}_{\mathbf{1\|3}}$	$\langle X\rangle$	$1\|0$	$0\|3$	$5\|0$	$0\|7$

Invariants for Lie superalgebras of $\mathrm{sdim} = (1|3)$.

Dimension 4: the classification. Lie superalgebras.

Theorem

The classification of 4-dimensional nilpotent Lie superalgebras over an algebraically closed field of characteristic different from 2 is given by:

$sdim(L) = (0|4)$: $L = \langle 0|x_1, x_2, x_3, x_4\rangle$

$\mathbf{L^1_{0|4}}$: $[\cdot,\cdot] = 0$.

$sdim(L) = (1|3)$: $L = \langle x_1|x_2, x_3, x_4\rangle$

$\mathbf{L^1_{1|3}}$ $(= \mathbf{L^a_{1|3}})$: *abelian;*

$\mathbf{L^2_{1|3}}$ $(= \mathbf{L^b_{1|3}})$: $[x_1, x_3] = x_4$;

$\mathbf{L^3_{1|3}}$ $(= \mathbf{L^c_{1|3}})$: $[x_2, x_3] = x_1$;

$\mathbf{L^4_{1|3}}$ $(= \mathbf{L^e_{1|3}})$: $[x_1, x_2] = x_3$, $[x_1, x_3] = x_4$;

$\mathbf{L^5_{1|3}}$ $(= \mathbf{L^f_{1|3}})$: $[x_3, x_3] = x_1$;

$\mathbf{L^6_{1|3}}$ $(= \mathbf{L^j_{1|3}})$: $[x_2, x_2] = x_1$, $[x_3, x_4] = x_1$.

$sdim(L) = (2|2)$: $L = \langle x_1, x_2|x_3, x_4\rangle$

$\mathbf{L^1_{2|2}}$ $(= \mathbf{L^a_{2|2}})$: *abelian;*

$\mathbf{L^2_{2|2}}$ $(= \mathbf{L^b_{2|2}})$: $[x_3, x_4] = x_2$;

$\mathbf{L^3_{2|2}}$ $(= \mathbf{L^e_{2|2}})$: $[x_3, x_3] = x_2$, $[x_3, x_4] = x_1$;

$\mathbf{L^4_{2|2}}$ $(= \mathbf{L^f_{2|2}})$: $[x_3, x_3] = [x_4, x_4] = x_2$, $[x_3, x_4] = x_1$;

$\mathbf{L^5_{2|2}}$ $(= \mathbf{L^g_{2|2}})$: $[x_1, x_3] = x_4$;

$\mathbf{L^6_{2|2}}$ $(= \mathbf{L^h_{2|2}})$: $[x_1, x_3] = x_4$, $[x_3, x_3] = x_2$.

$\mathbf{L^7_{2|2}}$ $(= \mathbf{L^i_{2|2}})$: $[x_4, x_4] = x_1$.

$sdim(L) = (3|1)$: $L = \langle x_1, x_2, x_3|x_4\rangle$

$\mathbf{L^1_{3|1}}$ $(= \mathbf{L^a_{3|1}})$: *abelian;*

$\mathbf{L^2_{3|1}}$ $(= \mathbf{L^b_{3|1}})$: $[x_1, x_2] = x_3$;

$\mathbf{L^3_{3|1}}$ $(= \mathbf{L^c_{3|1}})$: $[x_2, x_2] = x_3$;

$\mathbf{L^4_{3|1}}$ $(= \mathbf{L^d_{3|1}})$: $[x_1, x_2] = [x_3, x_4] = x_3$.

$sdim(L) = (4|0)$: $L = \langle x_1, x_2, x_3, x_4|0\rangle$

$\mathbf{L^1_{4|0}}$: *abelian;*

$\mathbf{L^2_{4|0}}$: $[x_1, x_2] = x_3$;

$\mathbf{L^3_{4|0}}$: $[x_1, x_2] = x_3$, $[x_1, x_3] = x_4$.

Dimension 4: the classification. $p|2p$ maps.

- Suppose that $L = L_{\bar{0}} \oplus L_{\bar{1}}$ is a p-nilpotent restricted Lie superalgebra. Then $L_{\bar{0}}$ is a p-nilpotent restricted Lie algebra with a p-map $(\cdot)^{[p]}$.

Dimension 4: the classification. $p|2p$ maps.

- Suppose that $L = L_{\bar{0}} \oplus L_{\bar{1}}$ is a p-nilpotent restricted Lie superalgebra. Then $L_{\bar{0}}$ is a p-nilpotent restricted Lie algebra with a p-map $(\cdot)^{[p]}$.
- The classification of 4-dimensional restricted Lie algebras has been achieved by Schneider-Usefi.

Dimension 4: the classification. $p|2p$ maps.

- Suppose that $L = L_{\bar{0}} \oplus L_{\bar{1}}$ is a p-nilpotent restricted Lie superalgebra. Then $L_{\bar{0}}$ is a p-nilpotent restricted Lie algebra with a p-map $(\cdot)^{[p]}$.
- The classification of 4-dimensional restricted Lie algebras has been achieved by Schneider-Usefi.
- We only have to check whether these p-maps satisfy

$$\mathrm{ad}_{e_i}^p(f_j) = \mathrm{ad}_{e_i^{[p]}}(f_j),$$

$\forall e_i$ basis elements of $L_{\bar{0}}$, $\forall f_i$ basis elements of $L_{\bar{1}}$.

Dimension 4: the classification. $p|2p$ maps.

Theorem

The p-nilpotent structures on nilpotent Lie superalgebras of total dimension 4 *with* $\dim(L_{\bar{1}}) > 0$ *are given by:*

- $sdim(L) = (0|4)$*: none.*
- $sdim(L) = (1|3)$*:* $x_1^{[p]} = 0$*.*
- $sdim(L) = (2|2)$*:*
  - $x_1^{[p]_1} = x_2^{[p]_1} = 0;$
  - $x_1^{[p]_2} = x_2,\ x_2^{[p]_2} = 0.$
- $sdim(L) = (3|1)$*:*
  - *Case* $L_{\bar{0}}$ *abelian:*
    - $x_1^{[p]_1} = x_2^{[p]_1} = x_3^{[p]_1} = 0;$
    - $x_1^{[p]_2} = x_2,\ x_2^{[p]_2} = x_3^{[p]_2} = 0.$
    - $x_1^{[p]_3} = x_2,\ x_2^{[p]_3} = x_3,\ x_3^{[p]_3} = 0.$
  - *Case* $L_{\bar{0}} \cong \mathbf{L}^{\mathbf{2}}_{\mathbf{3|0}} = \langle x_1, x_2, x_3; [x_1, x_2] = x_3 \rangle$*:*
    - $x_1^{[p]_4} = x_2^{[p]_4} = x_3^{[p]_4} = 0;$
    - $x_1^{[p]_5} = x_3,\ x_2^{[p]_5} = x_3^{[p]_5} = 0.$

Thank you for your attention!