Restricted Lie algebras in characteristic $p = 2$

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Non-associative algebras

A vector space $(A, +)$ over a field $\mathbb{K}$ is called **algebra** if it is endowed with a multiplicative law $A \times A \longrightarrow A$.

Sophus Lie
(1842-1899)

- **associative :** $a(bc) = (ab)c, \ \forall a, b, c \in A$;
- **commutative :** $ab = ba, \ \forall a, b \in A$;
- **Lie algebra :**
  1. $[a, b] = -[b, a], \ \forall a, b \in A$;
  2. $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0, \ \forall a, b, c \in A$ (Jacobi).

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Example : Let A be an associative algebra and $[a, b] := ab - ba, \ a, b \in A$.

Restricted Lie algebras

Let $\mathbb{K}$ a field of characteristic $p \geq 2$ and A an associative $\mathbb{K}$-algebra. With the commutator, it's a Lie algebra. The adjoint representation is then given by

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Let $m > 0$. Then

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Then, if $m = p$, we obtain

$$\mathrm{ad}_x^p(y) = x^p y - y x^p = \mathrm{ad}_{x^p}(y).$$

Restricted Lie algebras

Definition (Jacobson)

A **restricted Lie algebra** *is a Lie algebra L equipped with a map $(\cdot)^{[p]} : L \longrightarrow L$ satisfying for all $x, y \in L$ and for all $\lambda \in \mathbb{K}$:*

1. $(\lambda x)^{[p]} = \lambda^p x^{[p]}$;

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3. $(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x,y)$,

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with $is_i(x, y)$ the coefficient of Z^{i-1} in $\mathrm{ad}^{p-1}_{Zx+y}(x)$. Such a map $(\cdot)^{[p]} : L \longrightarrow L$ is called p-map.

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Example: any associative algebra A with $[a, b] = ab - ba$ and $a^{[p]} = a^p$, $\forall a, b \in A$.

Restricted Lie algebras

Definition

A Lie algebra morphism $f : \left(L, [\cdot,\cdot], (\cdot)^{[p]}\right) \to \left(L', [\cdot,\cdot]', (\cdot)^{[p]'}\right)$ *is called* **restricted** *if*

$$f\left(x^{[p]}\right) = f(x)^{[p]'}, \ \forall x \in L.$$

A L-module M is called **restricted** *if*

$$x^{[p]} \cdot m = \Big(\overbrace{x \cdot (x \cdots (x}^{p \text{ terms}} \cdot m) \cdots)\Big), \ \forall x \in L, \ \forall m \in M.$$

Restricted Lie algebras, $p = 2$

From now on, all the algebras will be considered over a field $\mathbb{F}$ of characteristic $p = 2$.

Definition

A **restricted Lie algebra** *in characteristic 2 is a Lie algebra L equipped with a map $(\cdot)^{[2]} : L \longrightarrow L$ such that, for all $x, y \in L$ and all $\lambda \in F$,*

1. $(\lambda x)^{[2]} = \lambda^2 x^{[2]}$;
2. $\left[x, y^{[2]}\right] = [[x, y], y]$;
3. $(x + y)^{[2]} = x^{[2]} + y^{[2]} + [x, y]$.

Proposition

Let L be a restricted Lie algebra in characteristic $p = 2$. Let $x_1, \cdots, x_n \in L$. Then we have the formula

$$\left(\sum_{i=1}^{n} x_i\right)^{[2]} = \sum_{i=1}^{n} x_i^{[2]} + \sum_{1 \le i < j \le n} [x_i, x_j].$$

Restricted Lie algebras, $p = 2$

Consider the formal space $L[[t]] := \left\{ \sum_i t^i x_i,\ x_i \in L \right\}$.

Proposition

Let L be a Lie algebra. Then $L[[t]]$ is a restricted Lie algebra with the extended bracket

$$\left[\sum_{i \geq 0} t^i x_i, \sum_{j \geq 0} t^j y_j \right] = \sum_{i,j} t^{i+j} [x_i, y_j], \quad \forall x_i, y_j \in L. \tag{1}$$

and the 2-mapping $(\cdot)^{[2]_t}$ given by

$$\left(\sum_{i \geq 0} t^i x_i \right)^{[2]_t} := \sum_{i \geq 0} t^{2i} x_i^{[2]} + \sum_{i,j} t^{i+j} [x_i, x_j]. \tag{2}$$

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Remark. By expanding the formula (2) and by arranging the terms by monomials of the same degree, we obtain

$$\left(\sum_{n\geq 0} t^n x_n\right)^{[2]_t} = \sum_{n\geq 0} t^n \left((\overline{n+1}) x_{\lfloor \frac{n}{2} \rfloor}^{[2]} + \sum_{\substack{i<j \\ i+j=n}} [x_i, x_j] \right), \tag{3}$$

where $\lfloor \cdot \rfloor$ denotes the floor function.

Chevalley-Eilenberg cohomology

Let L be an ordinary Lie algebra and M be an ordinary L-module. We set

$$C^m_{\mathrm{CE}}(L, M) = \mathrm{Hom}_{\mathbb{F}}(\Lambda^m L, M) \quad \text{for } m \geq 1,$$
$$C^0_{\mathrm{CE}}(L, M) \cong M.$$

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The differential maps $d^m_{\mathrm{CE}} : C^m_{\mathrm{CE}}(L, M) \longrightarrow C^{m+1}_{\mathrm{CE}}(L, M)$ are given by

$$\begin{aligned} d^m_{\mathrm{CE}}(\varphi)(x_1, \cdots, x_{m+1}) = & \sum_{1 \leq i < j \leq m+1} (-1)^{i+j-1} \varphi\big([x_i, x_j], x_1, \cdots \hat{x}_i, \cdots, \hat{x}_j, \cdots, x_{m+1}\big) \\ & + \sum_{i=1}^{m+1} (-1)^i x_i \varphi(x_1, \cdots, \hat{x}_i, \cdots, x_{m+1}), \end{aligned}$$

where $\hat{x}_i$ means that the element is omitted.

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$$+\sum_{i=1}^{m+1}(-1)^i x_i\varphi(x_1,\cdots,\hat{x}_i,\cdots,x_{m+1}),$$

where $\hat{x}_i$ means that the element is omitted.

Proposition

We have $d^{m+1}_{CE}\circ d^m_{CE} = 0.$

Restricted cohomology, $p = 2$

Let L be an restricted Lie algebra and M be a restricted L-module. A pair (φ, ω) with $\varphi : \wedge^n L \to M$ and $\omega : L^{n-1} \to M$ is a n-cochain if

1. $\omega(\lambda x, z_2, \cdots, z_{n-1}) = \lambda^2 \omega(x, z_2, \cdots, z_n), \ \lambda \in \mathbb{F};$

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2. ω is multilinear in $z_2, \cdots, z_{n-1}$;
3. $\omega(x + y, z_2, \cdots, z_{n-1}) =$ $\omega(x, z_2, \cdots, z_{n-1}) + \omega(y, z_2, \cdots, z_{n-1}) + \varphi(x, y, z_2, \cdots, z_{n-1})$.

We denote the spaces thus obtained by $C^n_{*_2}(L, M)$.

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We build the differential maps $d^n_{*_2} : C^n_{*_2}(L, M) \longrightarrow C^{n+1}_{*_2}(L, M)$.

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Let $d^n_{*_2}(\varphi, \omega) = (d^n_{CE}(\varphi), \delta^n(\omega))$, with

$$\begin{aligned}
\delta^n\omega(x, z_2, \cdots, z_n) &= x \cdot \varphi(x, z_2, \cdots, z_n) \\
&+ \sum_{i=2}^{n} z_i \cdot \omega(x, z_2, ..., \hat{z}_i, ..., z_n) \\
&+ \varphi(x^{[2]}, z_2, \cdots, z_n) \\
&+ \sum_{i=2}^{n} \varphi\left([x, z_i], x, z_2, \cdots, \hat{z}_i, \cdots, z_n\right) \\
&+ \sum_{1 \leq i < j \leq n} \omega\left(x, [z_i, z_j], z_2, \cdots, \hat{z}_i, \cdots, \hat{z}_i, \cdots, z_n\right).
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$$

$$
(\cdots) \longrightarrow C^{n-1}_{*_2}(L, M) \xrightarrow{d^{n-1}_{*2}} C^n_{*_2}(L, M) \xrightarrow{d^n_{*2}} C^{n+1}_{*_2}(L, M) \xrightarrow{d^{n+1}_{*2}} (\cdots)
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Restricted cohomology, $p = 2$

Proposition

1. *Let* $(\varphi, \omega) \in C^n_{*_2}(L, M)$. *Then* $(d^n_{CE}(\varphi), \delta^n(\omega)) \in C^{n+1}_{*_2}(L, M)$*;*

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2. *We have* $\delta^{n+1} \circ \delta^n = 0$.

We define:

$$Z^n_{*_2}(L, M) = \Big\{(\varphi, \omega) \in C^n_{*_2}(L, M), \ \ d^n_{\mathrm{CE}}(\varphi) = 0, \ \delta^n(\omega) = 0\Big\}.$$

$$B^n_{*_2}(L, M) = \Big\{(\varphi, \omega) \in C^n_{*_2}(L, M), \ \ (\varphi, \omega) \in \mathrm{im}(d^{n-1}_{\mathrm{CE}}, \delta^{n-1})\Big\}.$$

$$H^n_{*_2}(L, M) = Z^n_{*_2}(L, M)/B^n_{*_2}(L, M).$$

Formal Deformations

Let $\left(L, [\cdot,\cdot], (\cdot)^{[2]}\right)$ be a restricted Lie algebra.

Definition

A formal deformation of $\left(L, [\cdot,\cdot], (\cdot)^{[2]}\right)$ *is a restricted structure on the formal space* $L[[t]]$*, given for* $x, y \in L$*, by*

$$m_t : (x, y) \longmapsto [x, y] + \sum_{i \geq 1} t^i m_i(x, y), \quad \omega_t : x \longmapsto x^{[2]} + \sum_{j \geq 1} t^j \omega_j(x),$$

with $m_i : \wedge^2(L, L) \to L$ *and* $\omega_i : L \mapsto L$*. Moreover, the two following conditions must be satisfied, for* $x, y, z \in L$*:*

$$m_t\Big((x, m_t(y, z)\Big) + m_t\Big((y, m_t(z, x)\Big) + m_t\Big((z, m_t(x, y)\Big) = 0; \tag{4}$$

$$m_t\Big(x, \omega_t(y)\Big) = m_t\Big(m_t(\cdots m_t(x, \overbrace{y), y), \cdots, y}^{p \text{ terms}}\Big); \tag{5}$$

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A formal deformation is of order $N > 0$ if $m_t = \sum_{i\geq 0}^{N} t^i m_i, \;\; \omega_t = \sum_{j\geq 0}^{N} t^j \omega_j$.

Formal Deformations

Let $\left(L, [\cdot,\cdot], (\cdot)^{[2]}\right)$ be a restricted Lie algebra and $\left(L[[t]], m_t, \omega_t\right)$ a formal deformation.

Proposition

- *For all $i \geq 0$, we have $(m_i, \omega_i) \in C^2_{*_2}(L, L)$.*
- *The pair (m_1, ω_1) is a 2-cocycle, that is, $d^2_{CE}(m_1) = 0,\ \delta^n(\omega_1) = 0$.*

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For $N > 0$, we define for all $x, y, z \in L$ the quantities

$$\mathrm{obs}^{(1)}_{N+1}(x,y,z) = \sum_{i=1}^{N} \Big(m_i(x, m_{N+1-i}(y,z)) + m_i(y, m_{N+1-i}(z,x)) + m_i(z, m_{N+1-i}(x,y)) \Big);$$

$$\mathrm{obs}^{(2)}_{N+1}(x,y) = \sum_{i=1}^{N} \Big(m_i(y, \omega_{N+1-i}(x)) + m_i(m_{N+1-i}(y,x), x) \Big).$$

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Proposition

- *We have $\left(obs^{(1)}_{N+1}, obs^{(2)}_{N+1}\right) \in C^3_{*_2}(L, L)$.*
- *A deformation of order N extends to a deformation of order $N+1$ if and only if*

$$\left(obs^{(1)}_{N+1}, obs^{(2)}_{N+1}\right) \in B^3_{*_2}(L, L).$$

Formal Deformations

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Two formal deformations $\big(L[[t]], m_t, \omega_t\big)$ *and* $\big(L[[t]], m'_t, \omega'_t\big)$ *of a restricted Lie algebra are called equivalent if there is a formal automophism* $\phi_t = id + \sum_{i>0} t^i \phi_i$ *such that, for all* $x, y \in L$,

$$\phi_t\big(m_t(x, y)\big) = m\big(\phi_t(x), \phi_t(y)\big);$$
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Let $\big(L, [\cdot,\cdot], (\cdot)^{[2]}\big)$ *be a restricted Lie algebra and* $\big(L[[t]], m_t, \omega_t\big)$, $\big(L[[t]], m'_t, \omega'_t\big)$ *equivalent formal deformations. Then,* (m_1, ω_1) *and* (m'_1, ω'_1) *are in the same cohomological class.*

Example: Heisenberg algebras

Definition (Heisenberg algebra)

The three dimensional Heisenberg algebra $\mathcal{H}$ *is spanned by elements* x, y, z *and equipped with the Lie bracket* $[\cdot, \cdot]$ *defined by*

$$[x, y] = z, \ [x, z] = [y, z] = 0.$$

Werner Heisenberg (1901-1976)

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Proposition

There are up to isomorphism two restricted Heisenberg algebras in characteristic $p = 2$*, given by*

1. $x^{[2]} = y^{[2]} = z^{[2]} = 0$*, denoted by* $(\mathcal{H}, 0)$*;*
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Let $(\cdot)^{[2]}$ a 2-map on $\mathcal{H}$. Then,

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Let $u = ax + by + cz \in \mathcal{H}$, $a, b, c \in \mathbb{F}$. Then $u^{[2]} = a^2 x^{[2]} + b^2 y^{[2]} + c^2 z^{[2]} + abz$.

Restricted cohomology with adjoint coefficients

Theorem (Second cohomology group with adjoint coefficients, $p = 2$)

We have $\dim_{\mathbb{F}}\left(H^2_{*_2}(\mathcal{H}, 0)\right) = 3$ *and* $\dim_{\mathbb{F}}\left(H^2_{*_2}(\mathcal{H}, z^*)\right) = 2$.

- *A basis for* $H^2_{*_2}(\mathcal{H}, 0)$ *is given by* $\left\{(\varphi_1, \omega_1), (\varphi_2, \omega_2), (0, \omega_3)\right\}$, *with*

$$\varphi_1(y, z) = z;\ \varphi_2(x, z) = z;\ \omega_1(y) = y;\ \omega_2(x) = x;\ \omega_3(z) = z.$$

- *A basis for* $H^2_{*_2}(\mathcal{H}, z^*)$ *is given by* $\left\{(\varphi_1, \omega_1), (\varphi_2, \omega_2)\right\}$, *with*

$$\varphi_1(x, y) = x;\ \varphi_2(x, y) = y;\ \omega_1(y) = y;\ \omega_2(x) = x.$$

Thank you for your attention!