Lagrangian extensions of Lie superalgebras in characteristic 2

Quentin Ehret

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Joint work with Saïd Benayadi and Sofiane Bouarroudj

جامعة نيويورك أبوظبي
NYU | ABU DHABI

Outline of the talk

1. Lie superalgebras in characteristic 2
2. Left-symmetric superalgebras and connections
3. Lagrangian extensions of Lie superalgebras
4. Computation of 4-dimensional Lagrangian extensions

Lie superalgebras in characteristic 2, definition

A *Lie superalgebra* over a field $\mathbb{K}$ of characteristic $p = 2$ is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ such that:

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3. there is a map $s : \mathfrak{g}_{\bar{1}} \to \mathfrak{g}_{\bar{0}}$, satisfying $s(\lambda x) = \lambda^2 s(x)$, such that the bracket of two odd elements is given by:

$$[x, y] := s(x + y) - s(x) - s(y), \quad \forall x, y \in \mathfrak{g}_{\bar{1}}. \tag{1}$$

The Jacobi identity involving the squaring reads as follows:

$$[s(x), y] = [x, [x, y]], \ \forall x \in \mathfrak{g}_{\bar{1}}, \ \forall y \in \mathfrak{g}. \tag{2}$$

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Example: any associative superalgebra with $[a, b] = ab - ba$ and $s(a) = aa$.

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$$\mathfrak{g}^{(0)} = \mathfrak{g}, \quad \mathfrak{g}^{(i+1)} = [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}] + \mathrm{Span}\Big\{s(x),\ x \in (\mathfrak{g}^{(i)})_{\bar{1}}\Big\}.$$

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- Lie superalgebras in characteristic 2 admitting a Cartan matrix have been classified by Bouarroudj, Grozman, Leites, SIGMA 2009.

Lie superalgebras in characteristic 2, continued

Let $\mathfrak{g}$ and $\mathfrak{h}$ be two Lie superalgebras.

- An even linear map $\varphi : \mathfrak{g} \to \mathfrak{h}$ is called a *Lie superalgebras morphism* if

$$\begin{aligned} \varphi([x,y]_{\mathfrak{g}}) &= [\varphi(x),\varphi(y)]_{\mathfrak{h}}, \qquad && \forall x \in \mathfrak{g}_{\bar 0},\ \forall y \in \mathfrak{g}, \\ \varphi \circ s_{\mathfrak{g}}(x) &= s_{\mathfrak{h}} \circ \varphi(x), && \forall x \in \mathfrak{g}_{\bar 1}. \end{aligned}$$

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- A linear map $D : \mathfrak{g} \to \mathfrak{g}$ is called a *derivation* of $\mathfrak{g}$ if

$$\begin{aligned} D([x,y]) &= [D(x), y] + [x, D(y)], && \forall x \in \mathfrak{g}_{\bar{0}},\ y \in \mathfrak{g}, \\ D(s(x)) &= [D(x), x], && \forall x \in \mathfrak{g}_{\bar{1}}. \end{aligned}$$

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- A *representation* of $\mathfrak{g}$ in a $\mathbb{Z}/2\mathbb{Z}$-graded vector space M is an even map $\rho : \mathfrak{g} \to \mathrm{End}(M)$ satisfying

$$\begin{aligned} \rho([x, y]) &= [\rho(x), \rho(y)] \quad \forall x, y \in \mathfrak{g}; \\ \rho(s(x)) &= (\rho(x))^2 \quad \forall x \in \mathfrak{g}_{\bar{1}}. \end{aligned}$$

Lie superalgebras, classification in dim $= 2$

Proposition

Let $\mathfrak{g}$ be a 2-dimensional Lie superalgebras over an arbitrary field of characteristic 2. Then, $\mathfrak{g}$ is isomorphic to one of the following superalgebras.

- $\underline{sdim(\mathfrak{g}) = (0|2)}$: $\mathfrak{g} = \mathbf{L}^1_{0|2} = \langle 0 | e_1, e_2 \rangle$.
- $\underline{sdim(\mathfrak{g}) = (1|1)}$; $\mathfrak{g} = \langle e_1 | e_2 \rangle$.
  1. $\mathbf{L}^1_{1|1} = \langle e_1 | e_2; [e_1, e_2] = e_2 \rangle$;
  2. $\mathbf{L}^2_{1|1} = \langle e_1 | e_2; s(e_2) = e_1 \rangle$;
  3. $\mathbf{L}^3_{1|1} = \langle e_1 | e_2 \rangle$ (*abelian*);
- $\underline{sdim(\mathfrak{g}) = (2|0)}$: $\mathfrak{g} = \langle e_1, e_2 | 0 \rangle$.
  1. $\mathbf{L}^1_{2|0} = \langle e_1, e_2 | 0 \rangle$; $[e_1, e_2] = e_2$;
  2. $\mathbf{L}^2_{2|0} = \langle e_1, e_2 | 0 \rangle$ (*abelian*);

Lie superalgebras in characteristic 2, cohomology (1)

This cohomology was introduced by Bouarroudj and Makhlouf, Mathematics (2023).

Let $\mathfrak{g}$ be a Lie superalgebra in characteristic 2 and let M be a $\mathfrak{g}$-module.

A 1-cocycle on $\mathfrak{g}$ with values in M is a linear map $\varphi : \mathfrak{g} \to M$ such that

$$\begin{aligned} d^1_{\mathsf{CE}}(\varphi)(x,z) &:= x\cdot\varphi(z) + z\cdot\varphi(x) + \varphi([x,z]) = 0, \forall x, z \in \mathfrak{g}; \\ \delta^1(\varphi)(x) &:= x\cdot\varphi(x) + \varphi(s(x)) = 0, \qquad\qquad \forall x \quad \in \mathfrak{g}_{\bar{1}}. \end{aligned}$$

The space of 1-cocycles on $\mathfrak{g}$ with values in M is denoted by $XZ^1(\mathfrak{g}; M)$.
We also use the notation $\mathfrak{d}^1(\varphi) := (d^1_{\mathsf{CE}}(\varphi), \delta^1(\varphi))$.

Lie superalgebras in characteristic 2, cohomology (2)

A 2-cocycle on $\mathfrak{g}$ with values in M consists of a pair (α, γ) such that:

1. $\alpha : \mathfrak{g} \wedge \mathfrak{g} \to M$ is a bilinear map;
2. $\gamma : \mathfrak{g}_{\bar{1}} \to M$ satisfies

$$\gamma(\lambda x) = \lambda^2 \gamma(x), \qquad \forall \lambda \in \mathbb{K},\ \forall x \in \mathfrak{g};$$
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$$\alpha(x, y) = \gamma(x + y) + \gamma(x) + \gamma(y), \qquad \forall x, y \in \mathfrak{g}_{\bar{1}};$$
3. For all $x, y, z \in \mathfrak{g}$ and for all $u \in \mathfrak{g}_{\bar{1}}$, we have
$$d^2_{\mathrm{CE}}(\alpha)(x, y, z) := \underset{x,y,z}{\circlearrowleft} \left(x \cdot \alpha(y, z) + \alpha([x, y], z)\right) = 0;$$
$$\delta^2(\alpha, \gamma)(u, z) := u \cdot \alpha(u, z) + z \cdot \gamma(u) + \alpha(s(u), z) + \alpha([u, z], u) = 0.$$

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There is a graduation on $XZ^2(\mathfrak{g}; M)$ defined by $|(\alpha, \gamma)| := |\alpha|$.

Quasi-Frobenius Lie superalgebras

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra in characteristic 2. A bilinear form ω on $\mathfrak{g}$ with values in $\mathbb{K}$ is called

1. *ortho-orthogonal* if ω is even;
2. *periplectic* if ω is odd;
3. *closed* if the following cocycle conditions are satisfied:

$$\circlearrowleft_{x,y,z} \omega([x,y],z) = 0, \qquad \forall x,y,z \in \mathfrak{g}; \tag{3}$$

$$\omega(s(x),y) = \omega(x,[x,y]), \ \forall x \in \mathfrak{g}_{\bar{1}}, \ \forall y \in \mathfrak{g}. \tag{4}$$

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An *even* bilinear form on $\mathfrak{g}$ is called $\bar{1}$-*antisymmetric* if

$\omega(x,y) = \omega(y,x), \ \forall x,y \in \mathfrak{g}$ s. t. $|x| = |y|$; and $\omega(x,x) = 0 \ \forall x \in \mathfrak{g}_{\bar{0}}$.

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An *odd* bilinear form on $\mathfrak{g}$ is called $\bar{1}$-*antisymmetric* if

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A Lie superalgebra $\mathfrak{g}$ is called *quasi-Frobenius* if it is equipped with a $\bar{1}$-antisymmetric non-degenerate closed form ω.

Left-symmetric superalgebras in characteristic 2

A left-symmetric superalgebra $(V, \triangleright)$ in characteristic $p = 2$ is a vector superspace $V = V_{\bar{0}} \oplus V_{\bar{1}}$ endowed with a bilinear product $\triangleright : V \times V \to V$ satisfying

$$\begin{array}{llll} (i) & x \triangleright (y \triangleright z) + (x \triangleright y) \triangleright z & = & y \triangleright (x \triangleright z) + (y \triangleright x) \triangleright z, \quad \forall x, y, z \in V; \\ (ii) & x \triangleright (x \triangleright y) & = & (x \triangleright x) \triangleright y, \qquad \forall x \in V_{\bar{1}}, \ \forall y \in V. \end{array}$$

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Proposition

Let $(V, \triangleright)$ be a left-symmetric superalgebra. Then, $(\mathfrak{g}(V), [\cdot, \cdot], s)$ is a Lie superalgebra with $\mathfrak{g}(V) = V$ as superspaces and

$$[x, y] := x \triangleright y + y \triangleright x, \ \forall x \in V_{\bar{0}}, \forall y \in V; \tag{5}$$

$$s(x) := x \triangleright x, \quad \forall x \in V_{\bar{1}}. \tag{6}$$

A left-symmetric product $\triangleright$ on a Lie superalgebra $(V, [\cdot, \cdot], s)$ is called compatible with the Lie superalgebra structure if Conditions (5) and (6) are satisfied.

Left-symmetric superalgebras in characteristic 2, example

Proposition

Let $(\mathfrak{g}, [\cdot,\cdot], s)$ *be a Lie superalgebra equipped with an invertible derivation* D*. Let*

$$x \triangleright y := D^{-1}\Big([x, D(y)]\Big), \ \forall x, y \in \mathfrak{g}.$$

Then, $\triangleright$ *is a left-symmetric product compatible with the Lie structure.*

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Example. Consider the Hamiltonian superalgebra $\mathfrak{h}_\Pi(0|4)$ (see Benayadi and Bouarroudj, Journal of Algebra, 2018). As a vector space it can be considered as

$$\mathfrak{h}_\Pi(0|4) \simeq \mathrm{Span}\{H_f \mid f \in \mathbb{K}[\xi, \eta]\} \simeq [\xi, \eta] / \, \mathbb{K} \cdot 1,$$

where $\xi_1, \xi_2, \eta_1, \eta_2$ are odd indeterminates and

$$H_f = \frac{\partial f}{\partial \xi_1}\frac{\partial}{\partial \eta_1} + \frac{\partial f}{\partial \eta_1}\frac{\partial}{\partial \xi_1} + \frac{\partial f}{\partial \xi_2}\frac{\partial}{\partial \eta_2} + \frac{\partial f}{\partial \eta_2}\frac{\partial}{\partial \xi_2}.$$

The Lie bracket $[H_f, H_g] = H_{\{f,g\}}$ is given by the Poisson bracket:

$$\{f, g\} := \frac{\partial f}{\partial \xi_1}\frac{\partial g}{\partial \eta_1} + \frac{\partial f}{\partial \eta_1}\frac{\partial g}{\partial \xi_1} + \frac{\partial f}{\partial \xi_2}\frac{\partial g}{\partial \eta_2} + \frac{\partial f}{\partial \eta_2}\frac{\partial g}{\partial \xi_2}.$$

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Then, its simple derived superalgebra $\mathfrak{h}_\Pi^{(1)}(0|4)$ admits an invertible derivation, thus a left-symmetric structure.

Classification in dimension 2

For each Lie superalgebra $\mathfrak{g}$ of dimension 2, we classified up to isomorphism all the non-zero left-symmetric structures that are compatible with the bracket and the squaring of $\mathfrak{g}$.

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$\mathfrak{g}$	Bracket on $\mathfrak{g}$	Left-symmetric product on $\mathfrak{g}$	Conditions
$\mathbf{L}^{1}_{1\vert 1}$	$[e_1, e_2] = e_2$	$e_1 e_2 = e_2$	None
		$e_1 e_1 = \varepsilon e_1;\ e_1 e_2 = e_2$	$\varepsilon \neq 0, 1$
		$e_1 e_1 = e_1;\ e_1 e_2 = e_2$	None
		$e_1 e_1 = \varepsilon e_1;\ e_1 e_2 = (1+\varepsilon) e_2;\ e_2 e_1 = \varepsilon e_2$	$\varepsilon \neq 0$
$\mathbf{L}^{2}_{1\vert 1}$	$s(e_2) = e_1$	$e_2 e_2 = e_1$	None
		$e_1 e_1 = e_1;\ e_1 e_2 = e_2;\ e_2 e_1 = e_2;\ e_2 e_2 = e_1$	None
$\mathbf{L}^{3}_{1\vert 1}$	abelian	$e_1 e_1 = e_1$	None
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The case where $\mathrm{sdim}(\mathfrak{g}) = (1\vert 1)$.

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		$e_1 e_1 = e_1;\ e_1 e_2 = e_2;\ e_2 e_1 = e_2;\ e_2 e_2 = e_1$	None
$\mathbf{L}^3_{1\vert 1}$	abelian	$e_1 e_1 = e_1$	None
		$e_1 e_1 = e_1;\ e_1 e_2 = e_2; e_2 e_1 = e_2$	None

The case where $\mathrm{sdim}(\mathfrak{g}) = (1|1)$.

- $\mathfrak{g} = \mathbf{L}^1_{2|0}$, $[e_1, e_2] = e_2$: 10 non-isomorphic left-symmetric products;
- $\mathfrak{g} = \mathbf{L}^2_{2|0}$, abelian: 5 non-isomorphic left-symmetric products.

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The **curvature of the connection** ∇ is given by a pair of maps (R, S), where $R : \mathfrak{g} \times \mathfrak{g} \to \mathrm{End}(\mathfrak{g}),$ and $S : \mathfrak{g}_{\bar{1}} \to \mathrm{End}(\mathfrak{g}),$ are defined by

$$\begin{aligned} R(x, y) &:= \nabla_{[x,y]} + [\nabla_x, \nabla_y], \ \forall x, y \in \mathfrak{g}; \\ S(x) &:= \nabla_{s(x)} + \nabla_x^2, \qquad \forall x \in \mathfrak{g}_{\bar{1}}. \end{aligned}$$

The connection ∇ is called *flat* if $(R, S) = (0, 0)$.

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As in characteristic zero, a flat connection on $\mathfrak{g}$ whose covariant derivative of the torsion vanishes defines a (characteristic 2 version) of a **post-Lie** product.

Lagrangian extensions

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- **Our goal:** the characteristic 2 case.

Change of parity, dual representations

Change of parity functor

Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ a $\mathbb{Z}_2$-graded vector space. We denote by Π the *change of parity functor* $\Pi : V \mapsto \Pi(V)$, where $\Pi(V)$ is another copy of V such that

$$\Pi(V)_{\bar{0}} := V_{\bar{1}}; \quad \Pi(V)_{\bar{1}} := V_{\bar{0}}.$$

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Let $(\mathfrak{g}, \omega)$ be a quasi-Frobenius Lie superalgebra. A **strong polarization** of $(\mathfrak{g}, \omega)$ is a decomposition $\mathfrak{g} = \mathfrak{a} \oplus N$ as vector superspaces, where $\mathfrak{a}$ is a homogeneous Lagrangian ideal of $\mathfrak{g}$ ($\mathfrak{a}^{\perp} = \mathfrak{a}$) and N is a Lagrangian subspace.

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Let $(\mathfrak{h}, [\cdot, \cdot]_{\mathfrak{h}}, s_{\mathfrak{h}})$ be a Lie superalgebra endowed with a torsion-free flat connection $\nabla : \mathfrak{h} \rightarrow \mathrm{End}(\mathfrak{h})$. We define the *dual representations*

$$\rho : \mathfrak{h} \rightarrow \mathrm{End}(\mathfrak{h}^*) \text{ and } \chi : \mathfrak{h} \rightarrow \mathrm{End}(\Pi(\mathfrak{h}^*)),$$

$$\begin{aligned} \rho(x)(\xi) &:= \xi \circ \nabla_x, & \forall x \in \mathfrak{h}, \ \forall \xi \in \mathfrak{h}^*; \\ \chi(x)\Big(\Pi(\xi)\Big) &:= \Pi \circ \rho(x) \circ \Pi\Big(\Pi(\xi)\Big). \end{aligned}$$

Construction of the Lagrangian extensions

Let $(\alpha, \gamma) \in XZ^2(\mathfrak{h}, \mathfrak{h}^*)_{\bar{0}}$ (resp. $(\beta, \theta) \in XZ^2(\mathfrak{h}, \Pi(\mathfrak{h}^*))_{\bar{0}}$) be 2-cocycles.

On the space $\mathfrak{g} := \mathfrak{h} \oplus \mathfrak{h}^*$. The brackets and squaring are defined as follows:

$$[x, y]_{\mathfrak{g}} := [x, y]_{\mathfrak{h}} + \alpha(x, y), \quad [x, \xi]_{\mathfrak{g}} := \rho(x)(\xi), \ \forall x, y \in \mathfrak{h}, \ \forall \xi \in \mathfrak{h}^*.$$

$$s_{\mathfrak{g}}(x + \xi) := s_{\mathfrak{h}}(x) + \gamma(x) + \rho(x)(\xi), \quad \forall x \in \mathfrak{h}_{\bar{1}}, \ \forall \xi \in \mathfrak{h}^*_{\bar{1}}.$$

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$$s_{\mathfrak{g}}(x+\xi):=s_{\mathfrak{h}}(x)+\gamma(x)+\rho(x)(\xi),\quad \forall x\in\mathfrak{h}_{\bar 1},\ \forall\xi\in\mathfrak{h}^*_{\bar 1}.$$

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$$\omega(x+\xi,y+\zeta)=\xi(y)+\zeta(x),\quad \forall x+\xi,y+\zeta\in\mathfrak{g}.$$

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$$[x, y]_{\mathfrak{g}} := [x, y]_{\mathfrak{h}} + \beta(x, y), \quad [x, \Pi(\xi)]_{\mathfrak{g}} := \chi(x)(\Pi(\xi)), \quad \forall x, y \in \mathfrak{h}, \ \forall \Pi(\xi) \in \Pi(\mathfrak{h}^*).$$

$$s_{\mathfrak{g}}(x) := s_{\mathfrak{h}}(x) + \theta(x) + \chi(x)(\Pi(\xi)), \quad \forall x \in \mathfrak{h}_{\bar{1}}, \ \forall \Pi(\xi) \in \Pi(\mathfrak{h}^*_{\bar{1}}).$$

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We define a periplectic form as follows:

$$\kappa(x + \Pi(\xi), y + \Pi(\zeta)) = \xi(y) + \zeta(x), \quad \forall x + \xi, y + \zeta \in \mathfrak{g}.$$

Construction of the Lagrangian extensions

In the sequel, I will focus on the ortho-orthogonal case.

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We define the first and second Lagrangian cochain spaces as

$$XC^1_L(\mathfrak{h},\mathfrak{h}^*) := \Big\{\psi \in XC^1(\mathfrak{h},\mathfrak{h}^*),\ \psi(x)(y) = \psi(y)(x),\ \forall x,y \in \mathfrak{h}\Big\},$$
$$XC^2_L(\mathfrak{h},\mathfrak{h}^*) := \Big\{(\alpha,\gamma) \in XC^2(\mathfrak{h},\mathfrak{h}^*),\ \text{satisfying (7) and (8)}\Big\},\ \text{where}$$

$$\underset{x,y,z}{\circlearrowleft} \alpha(x,y)(z) = 0,\ \forall x,y,z \in \mathfrak{h}; \tag{7}$$
$$\gamma(x)(y) + \alpha(x,y)(x) = 0,\ \forall x \in \mathfrak{h}_{\bar{1}},\ \forall y \in \mathfrak{h}. \tag{8}$$

Construction of the Lagrangian extensions

Theorem

Let $(\mathfrak{h}, \nabla)$ be a Lie superalgebra equipped with a flat and torsion-free connection ∇ and let $(\alpha, \gamma) \in XZ^2(\mathfrak{h}, \mathfrak{h}^*)_{\bar{0}}$ be an even 2-cocycle.

1. *The form ω on $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^*$ is closed if and only if (α, γ) is a* [*Lagrangian*] *2-cocycle.*

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1. *The form ω on $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^*$ is closed if and only if (α, γ) is a* $\boxed{\textit{Lagrangian}}$ *2-cocycle.*
2. *In this case, one can canonically define a strongly polarized quasi-Frobenius Lie superalgebra $(\mathfrak{g}, \omega, \mathfrak{h}^*, \mathfrak{h})$, where $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^*$, called T^*-extension of $(\mathfrak{h}, \nabla)$.*

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Let $(\mathfrak{g}, \omega_{\mathfrak{g}}, \mathfrak{a}, N)$ *be a strongly polarized quasi-Frobenius Lie superalgebra and let* $\mathfrak{h} := \mathfrak{g}/\,\mathfrak{a}$.

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$$\omega_{\mathfrak{h}}(u, a) := \omega_{\mathfrak{g}}(\tilde{u}, a), \quad \forall u \in \mathfrak{h}, \ \forall a \in \mathfrak{a}, \tag{9}$$

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where $\tilde{u}$ *is a lift of* u *to* $\mathfrak{g}$.
2. *There is a flat and torsion-free connection* ∇ *on* $\mathfrak{h}$ *defined by*
$$\omega_{\mathfrak{h}}(\nabla_u(v), a) = \omega_{\mathfrak{g}}(\tilde{v}, [\tilde{u}, a]), \; \forall u, v \in \mathfrak{h}, \; \forall a \in \mathfrak{a}\,. \tag{10}$$

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3. *If ω is ortho-orthogonal, there exists an even Lagrangian cocycle (α, γ) such that $(\mathfrak{g}, \omega, \mathfrak{a}, N)$ is isomorphic to the T^*-extension of $(\mathfrak{h}, \nabla)$ by (α, γ).*

Equivalence of Lagrangian extensions

An *isomorphism of Lagrangian extensions of* $\mathfrak{h}$ is a Lie isomorphism $\Phi : (\mathfrak{g}, \omega) \to (\mathfrak{g}', \omega')$ satisfying $\omega(x, y) = \omega'\Big(\Phi(x), \Phi(y)\Big), \ \forall x, y \in \mathfrak{g}$, such that the following diagram commutes:

$$
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0 & \longrightarrow & \mathfrak{a} & \longrightarrow & \mathfrak{g} & \longrightarrow & \mathfrak{h} & \longrightarrow & 0 \\
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Theorem

Let $(\mathfrak{h}, \nabla)$ be a Lie superalgebra equipped with a flat torsion-free connection ∇. Two Lagrangian extensions of $(\mathfrak{h}, \nabla)$ are isomorphic if and only if they correspond to the same extension 2-cocycle in $XH^2_L(\mathfrak{h}, \mathfrak{h}^*)_{\bar{0}}$.

An example

Consider $\mathfrak{h} := \mathbf{L}^{\mathbf{1}}_{\mathbf{1}|\mathbf{1}}$ given in the basis $(e|f)$ by $[e, f] = f$ and $s = 0$. Let $\varepsilon \in \mathbb{K}$. We define a flat torsion-free connection ∇^{ε} on $\mathfrak{h}$ by

$$\nabla^{\varepsilon}_{e}(e) = (1+\varepsilon)e, \quad \nabla^{\varepsilon}_{e}(f) = \varepsilon f, \quad \nabla^{\varepsilon}_{f}(e) = (1+\varepsilon)f, \quad \nabla^{\varepsilon}_{f}(f) = 0.$$

1. In the case where $\varepsilon \neq 1$, we have $XH^2(\mathfrak{h}, \mathfrak{h}^*) = 0$;

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1. In the case where $\varepsilon \neq 1$, we have $XH^2(\mathfrak{h},\mathfrak{h}^*) = 0$;
2. In the case where $\varepsilon = 1$, $XH^2_L(\mathfrak{h},\mathfrak{h}^*)$ is one-dimensional and spanned by (α_2, γ_3^1), where

$$\alpha_2 = f^* \otimes e^* \wedge f^*; \quad \gamma_3^1(f) = e^*.$$

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6. Some of them are isomorphic or symplectomorphic. We detect them, using some tables of invariants that we computed.

Thank you for your attention

Main reference:

S. Benayadi, S. Bouarroudj, Q. Ehret,
Left-symmetric superalgebras and Lagrangian extensions of Lie superalgebras in characteristic 2,
to appear on arXiv.