Central extensions of restricted Lie superalgebras and classification of p-nilpotent Lie superalgebras in dimension 4

Quentin Ehret

Strasbourg, January 2025

Joint work with Sofiane Bouarroudj

جامعة نيويورك أبوظبي
NYU ABU DHABI

Restricted Lie algebras

Let $\mathbb{K}$ a field of characteristic $p > 2$ and A an associative $\mathbb{K}$-algebra. With the commutator, it's a Lie algebra. The adjoint representation is then given by

$$\mathrm{ad}_x(y) = xy - yx.$$

Restricted Lie algebras

Let $\mathbb{K}$ a field of characteristic $p > 2$ and A an associative $\mathbb{K}$-algebra. With the commutator, it's a Lie algebra. The adjoint representation is then given by

$$\mathrm{ad}_x(y) = xy - yx.$$

Let $m > 0$. Then

$$\mathrm{ad}_x^m(y) = \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} x^j y x^{m-j}.$$

Restricted Lie algebras

Let $\mathbb{K}$ a field of characteristic $p > 2$ and A an associative $\mathbb{K}$-algebra. With the commutator, it's a Lie algebra. The adjoint representation is then given by

$$\mathrm{ad}_x(y) = xy - yx.$$

Let $m > 0$. Then

$$\mathrm{ad}_x^m(y) = \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} x^j y x^{m-j}.$$

Then, if $m = p$, we obtain

$$\mathrm{ad}_x^p(y) = x^p y - y x^p = \mathrm{ad}_{x^p}(y).$$

Restricted Lie algebras

Definition (Jacobson)

A **restricted Lie algebra** *is a Lie algebra L equipped with a map $(\cdot)^{[p]} : L \longrightarrow L$ satisfying for all $x, y \in L$ and for all $\lambda \in \mathbb{K}$:*

1. $(\lambda x)^{[p]} = \lambda^p x^{[p]}$;

Nathan Jacobson (1910-1999)

Restricted Lie algebras

Definition (Jacobson)

A **restricted Lie algebra** *is a Lie algebra L equipped with a map $(\cdot)^{[p]} : L \longrightarrow L$ satisfying for all $x, y \in L$ and for all $\lambda \in \mathbb{K}$:*

1. $(\lambda x)^{[p]} = \lambda^p x^{[p]}$;
2. $\left[x, y^{[p]}\right] = [[\cdots[x, \overbrace{y], y], \cdots, y}^{p \text{ terms}}]$;

Nathan Jacobson (1910-1999)

Restricted Lie algebras

Definition (Jacobson)

A **restricted Lie algebra** *is a Lie algebra L equipped with a map $(\cdot)^{[p]} : L \longrightarrow L$ satisfying for all $x, y \in L$ and for all $\lambda \in \mathbb{K}$:*

1. $(\lambda x)^{[p]} = \lambda^p x^{[p]}$;
2. $\left[x, y^{[p]}\right] = [[\cdots[x, \overbrace{y], y], \cdots, y}^{p \text{ terms}}]$;
3. $(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x, y)$,

Nathan Jacobson (1910-1999)

with is$_i(x, y)$ the coefficient of Z^{i-1} in $\mathrm{ad}^{p-1}_{Zx+y}(x)$. Such a map $(-)^{[p]} : L \longrightarrow L$ is called p-map.

Restricted Lie algebras

Definition (Jacobson)

A **restricted Lie algebra** *is a Lie algebra L equipped with a map $(\cdot)^{[p]} : L \longrightarrow L$ satisfying for all $x, y \in L$ and for all $\lambda \in \mathbb{K}$:*

1. $(\lambda x)^{[p]} = \lambda^p x^{[p]}$;
2. $\left[x, y^{[p]}\right] = [[\cdots[x, \overbrace{y], y], \cdots, y}^{p \text{ terms}}]$;
3. $(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x,y)$,

Nathan Jacobson (1910-1999)

with is$_i(x,y)$ the coefficient of Z^{i-1} in $\mathrm{ad}^{p-1}_{Zx+y}(x)$. Such a map $(-)^{[p]} : L \longrightarrow L$ is called p-map.

Example: any associative algebra A with $[a,b] = ab - ba$ and $a^{[p]} = a^p$, $\forall a, b \in A$.

Restricted Lie algebras

Very useful :

$$\sum_{i=1}^{p-1} s_i(x,y) = \sum_{\substack{x_i = x \text{ or } y \\ x_p = x,\ x_{p-1} = y}} \frac{1}{\sharp\{x\}} [x_1,[x_2,[\ldots,[x_{p-1},x_p]\ldots],$$

Restricted Lie algebras

Very useful :

$$\sum_{i=1}^{p-1} s_i(x,y) = \sum_{\substack{x_i = x \text{ or } y \\ x_p = x,\ x_{p-1} = y}} \frac{1}{\sharp\{x\}} [x_1, [x_2, [\ldots, [x_{p-1}, x_p]\ldots],$$

Definition

A Lie algebra morphism $f : \left(L, [\cdot,\cdot], (\cdot)^{[p]}\right) \to \left(L', [\cdot,\cdot]', (\cdot)^{[p]'}\right)$ *is called* **restricted** *if*

$$f\left(x^{[p]}\right) = f(x)^{[p]'},\ \forall x \in L.$$

A L-module M is called **restricted** *if*

$$x^{[p]} \cdot m = \Big(\overbrace{x \cdot (x \cdots (x}^{p \text{ terms}} \cdot m) \cdots)\Big),\ \forall x \in L,\ \forall m \in M.$$

Lie superalgebras

Definition

A Lie superalgebra $L = L_{\bar{0}} \oplus L_{\bar{1}}$ *is a* $\mathbb{Z}/2\mathbb{Z}$*-graded vector space equipped with a bilinear map* $[\cdot,\cdot] : L \times L \to L$ *satisfying for* $x, y, z \in L$ *:*

1. $|[x,y]| = |x| + |y|$;
2. $[x,y] = -(-1)^{|x||y|}[y,x]$;
3. $(-1)^{|x||z|}[x,[y,z]] + (-1)^{|x||y|}[y,[z,x]] + (-1)^{|y||z|}[z,[x,y]] = 0.$

If $p = 3$, the identity $[x,[x,x]] = 0,\ x \in L_{\bar{1}}$ has to be added as an axiom as well.

Lie superalgebras

Definition

A Lie superalgebra $L = L_{\bar{0}} \oplus L_{\bar{1}}$ is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space equipped with a bilinear map $[\cdot,\cdot] : L \times L \to L$ satisfying for $x, y, z \in L$:

1. $|[x,y]| = |x| + |y|$;
2. $[x,y] = -(-1)^{|x||y|}[y,x]$;
3. $(-1)^{|x||z|}[x,[y,z]] + (-1)^{|x||y|}[y,[z,x]] + (-1)^{|y||z|}[z,[x,y]] = 0.$

If $p = 3$, the identity $[x,[x,x]] = 0,\ x \in L_{\bar{1}}$ has to be added as an axiom as well.

Let $f : V \to W$ be a map between $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces. Then:

- the map f is called **even** if $f(V_{\bar{i}}) \subset W_{\bar{i}}$;
- the map f is called **odd** if $f(V_{\bar{i}}) \subset W_{\overline{i+1}}$;

Restricted Lie superalgebras

Definition (Restricted Lie superalgebra)

A **restricted Lie superalgebra** *is a Lie superalgebra* $L = L_{\bar{0}} \oplus L_{\bar{1}}$ *such that*

1. *The even part* $L_{\bar{0}}$ *is a restricted Lie algebra;*
2. *The odd part* $L_{\bar{1}}$ *is a Lie* $L_{\bar{0}}$*-module;*
3. $[x, y^{[p]}] = [[\ldots[x, \overbrace{y], y], \ldots, y}^{p \text{ terms}}], \ \forall x \in L_{\bar{1}}, \ y \in L_{\bar{0}}.$

Restricted Lie superalgebras

Definition (Restricted Lie superalgebra)

A **restricted Lie superalgebra** *is a Lie superalgebra* $L = L_{\bar{0}} \oplus L_{\bar{1}}$ *such that*

1. *The even part* $L_{\bar{0}}$ *is a restricted Lie algebra;*
2. *The odd part* $L_{\bar{1}}$ *is a Lie* $L_{\bar{0}}$*-module;*
3. $[x, y^{[p]}] = [[\ldots[x, \overbrace{y], y], \ldots, y}^{p \text{ terms}}], \ \forall x \in L_{\bar{1}}, \ y \in L_{\bar{0}}.$

We can define a map $(\cdot)^{[2p]} : L_{\bar{1}} \to L_{\bar{0}}$ *by*

$$x^{[2p]} = \left(x^2\right)^{[p]}, \ \text{with } x^2 = \frac{1}{2}[x, x], \ x \in L_{\bar{1}}.$$

Restricted Lie superalgebras

Definition (Restricted Lie superalgebra)

A **restricted Lie superalgebra** *is a Lie superalgebra* $L = L_{\bar{0}} \oplus L_{\bar{1}}$ *such that*

1. *The even part* $L_{\bar{0}}$ *is a restricted Lie algebra;*
2. *The odd part* $L_{\bar{1}}$ *is a Lie* $L_{\bar{0}}$*-module;*
3. $\left[x, y^{[p]}\right] = [[\ldots[x, \overbrace{y], y], \ldots, y}^{p \text{ terms}}], \ \forall x \in L_{\bar{1}}, \ y \in L_{\bar{0}}.$

We can define a map $(\cdot)^{[2p]} : L_{\bar{1}} \to L_{\bar{0}}$ *by*

$$x^{[2p]} = \left(x^2\right)^{[p]}, \ \text{with } x^2 = \frac{1}{2}[x, x], \ x \in L_{\bar{1}}.$$

Theorem (Jacobson)

Let $(e_j)_{j \in J}$ *be a basis of* $L_{\bar{0}}$*, and let the elements* $f_j \in L_{\bar{0}}$ *be such that* $(\mathrm{ad}_{e_j})^p = \mathrm{ad}_{f_j}$*. Then, there exists exactly one* $p|2p$*-mapping* $(\cdot)^{[p|2p]} : L \to L$ *such that*

$$e_j^{[p]} = f_j \quad \text{for all } j \in J.$$

Chevalley-Eilenberg cohomology for Lie superalgebras

Let $L = L_{\bar{0}} \oplus L_{\bar{1}}$ be a restricted Lie superalgebra and let $M = M_{\bar{0}} \oplus M_{\bar{1}}$ be a restricted module.
For $n = 0$: $C^0_{\mathrm{CE}}(L, M) := M$.
For $n > 0$: $C^n_{\mathrm{CE}}(L, M)$ is the space of n-linear super anti-symmetric maps with values in M.

Chevalley-Eilenberg cohomology for Lie superalgebras

Let $L = L_{\bar{0}} \oplus L_{\bar{1}}$ be a restricted Lie superalgebra and let $M = M_{\bar{0}} \oplus M_{\bar{1}}$ be a restricted module.
For $n = 0$: $C^0_{\mathsf{CE}}(L, M) := M$.
For $n > 0$: $C^n_{\mathsf{CE}}(L, M)$ is the space of n-linear super anti-symmetric maps with values in M.

$$d^0_{\mathrm{CE}}(m)(x) = (-1)^{|m||x|} x \cdot m \quad \forall m \in M \text{ and } \forall x \in L;$$

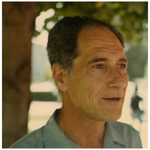
$$\begin{aligned} & d^n_{\mathrm{CE}}(\varphi)(x_1, \ldots, x_n) \\ &= \sum_{i<j} (-1)^{|x_j|(|x_{i+1}|+\cdots+|x_{j-1}|)+j} \varphi(x_1, \ldots, x_{i-1}, [x_i, x_j], x_{i+1} \ldots, \widetilde{x}_j, \ldots, x_n) \\ &+ \sum_j (-1)^{|x_j|(|\varphi|+|x_1|+\cdots+|x_{j-1}|)+j} x_j \cdot \varphi(x_1, \ldots, \widetilde{x}_j, \ldots, x_n) \end{aligned}$$

for any $\varphi \in C^{n-1}_{\mathsf{CE}}(L; M)$ with $n > 0$, and $x_1, \ldots, x_n \in L$.

The spaces $C^n_{\mathsf{CE}}(L; M)$ are $\mathbb{Z}_2$-graded.

A (very) brief history of restricted cohomology

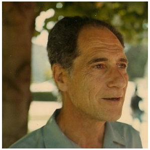
- 1955 (Hochschild): $H_*^n(L, M) := \mathrm{Ext}^n_{U_p(L)}(\mathbb{F}, M)$.



Gerhard Hochschild

A (very) brief history of restricted cohomology

- 1955 (Hochschild): $H^n_*(L, M) := \mathrm{Ext}^n_{U_p(L)}(\mathbb{F}, M)$.


Gerhard Hochschild

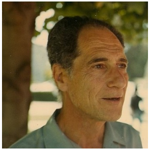
- 2000 (Evans-Fuchs): explicit constructions of 2-cocycles and central extensions.


Tyler J. Evans


Dmitry B. Fuchs

A (very) brief history of restricted cohomology

- 1955 (Hochschild): $H^n_*(L, M) := \mathrm{Ext}^n_{U_p(L)}(\mathbb{F}, M)$.


Gerhard Hochschild

- 2000 (Evans-Fuchs): explicit constructions of 2-cocycles and central extensions.


Tyler J. Evans


Dmitry B. Fuchs

- 2020 (Yuan-Chen-Cao): attempt to generalize to the superalgebras case.

Restricted cohomology for restricted Lie superalgebras

Let $L = L_{\bar{0}} \oplus L_{\bar{1}}$ be a restricted Lie superalgebra and let M be a L-supermodule.

We set $C^0_*(L, M) = M$ and $C^1_*(L, M) = \text{Hom}(L, M)$.

Restricted cohomology for restricted Lie superalgebras

Let $L = L_{\bar{0}} \oplus L_{\bar{1}}$ be a restricted Lie superalgebra and let M be a L-supermodule.

We set $C^0_*(L, M) = M$ and $C^1_*(L, M) = \mathrm{Hom}(L, M)$.

Definition (Restricted 2-cochains)

*Let $\varphi \in C^2_{CE}(L, M)$ (ordinary Chevalley-Eilenberg 2-cochain) and $\omega : L \longrightarrow M$. Then ω is φ-**compatible** if*

1. $\omega(\lambda x) = \lambda^p \omega(x), \ \lambda \in \mathbb{F}, \ x \in L_{\bar{0}};$

Restricted cohomology for restricted Lie superalgebras

Let $L = L_{\bar{0}} \oplus L_{\bar{1}}$ be a restricted Lie superalgebra and let M be a L-supermodule.

We set $C^0_*(L, M) = M$ and $C^1_*(L, M) = \mathrm{Hom}(L, M)$.

Definition (Restricted 2-cochains)

*Let $\varphi \in C^2_{CE}(L, M)$ (ordinary Chevalley-Eilenberg 2-cochain) and $\omega : L \longrightarrow M$. Then ω is φ-**compatible** if*

1. $\omega(\lambda x) = \lambda^p \omega(x),\ \lambda \in \mathbb{F},\ x \in L_{\bar{0}}$;
2. $\omega(x + y) =$

$$\omega(x)+\omega(y)+\sum_{\substack{x_i=x \text{ or } y\\ x_1=x,\ x_2=y}} \frac{1}{\pi(x)} \sum_{k=0}^{p-2} (-1)^k x_p \dots x_{p-k+1} \varphi\big([[\dots[x_1, x_2], x_3]\dots, x_{p-k-1}], x_{p-k}\big),$$

with $x, y \in L$, $\pi(x)$ the number of factors x_i equal to x.

Restricted cohomology for restricted Lie superalgebras

Let $L = L_{\bar{0}} \oplus L_{\bar{1}}$ be a restricted Lie superalgebra and let M be a L-supermodule.

We set $C^0_*(L, M) = M$ and $C^1_*(L, M) = \mathrm{Hom}(L, M)$.

Definition (Restricted 2-cochains)

*Let $\varphi \in C^2_{CE}(L, M)$ (ordinary Chevalley-Eilenberg 2-cochain) and $\omega : L \longrightarrow M$. Then ω is φ-**compatible** if*

1. $\omega(\lambda x) = \lambda^p \omega(x),\ \lambda \in \mathbb{F},\ x \in L_{\bar{0}};$
2. $\omega(x + y) =$

$$\omega(x) + \omega(y) + \sum_{\substack{x_i = x \text{ or } y \\ x_1 = x,\ x_2 = y}} \frac{1}{\pi(x)} \sum_{k=0}^{p-2} (-1)^k x_p \ldots x_{p-k+1} \varphi\big([[\ldots[x_1, x_2], x_3] \ldots, x_{p-k-1}], x_{p-k}\big),$$

with $x, y \in L$, $\pi(x)$ the number of factors x_i equal to x.

$$C^2_*(L, M) = \big\{(\varphi, \omega),\ \varphi \in C^2_{CE}(L, M),\ \omega \text{ is } \varphi\text{-compatible}\big\}$$

$\rightsquigarrow$ We have a similar (although more complicated) definition for $C^3_*(L, M)$.

Restricted cohomology for restricted Lie superalgebras

For $(\varphi, \omega) \in C^2_*(L; M)$, we write

$$(\varphi, \omega) = (\varphi_{\bar{0}}, \omega_{\bar{0}}) + (\varphi_{\bar{1}}, \omega_{\bar{1}}), \text{ where } \mathrm{Im}(\omega_{\bar{j}}) \subseteq M_{\bar{j}}. \tag{1}$$

Observe that also $(\varphi_{\bar{j}}, \omega_{\bar{j}}) \in C^2_*(L; M)$, thanks to the φ-compatibility.

Restricted cohomology for restricted Lie superalgebras

For $(\varphi, \omega) \in C^2_*(L; M)$, we write

$$(\varphi, \omega) = (\varphi_{\bar{0}}, \omega_{\bar{0}}) + (\varphi_{\bar{1}}, \omega_{\bar{1}}), \text{ where } \mathrm{Im}(\omega_{\bar{j}}) \subseteq M_{\bar{j}}. \quad (1)$$

Observe that also $(\varphi_{\bar{j}}, \omega_{\bar{j}}) \in C^2_*(L; M)$, thanks to the φ-compatibility.

In the sequel we will define the maps

$$0 \longrightarrow \underset{(=C^0_{\mathrm{CE}}(L,M))}{C^0_*(L, M)} \xrightarrow{d^0_*} \underset{(=C^1_{\mathrm{CE}}(L,M))}{C^1_*(L, M)} \xrightarrow{d^1_*} C^2_*(L, M) \xrightarrow{d^2_*} C^3_*(L, M).$$

Restricted cohomology for restricted Lie superalgebras

For $(\varphi, \omega) \in C^2_*(L; M)$, we write

$$(\varphi, \omega) = (\varphi_{\bar{0}}, \omega_{\bar{0}}) + (\varphi_{\bar{1}}, \omega_{\bar{1}}), \text{ where } \operatorname{Im}(\omega_{\bar{j}}) \subseteq M_{\bar{j}}. \quad (1)$$

Observe that also $(\varphi_{\bar{j}}, \omega_{\bar{j}}) \in C^2_*(L; M)$, thanks to the φ-compatibility.

In the sequel we will define the maps

$$0 \longrightarrow \underset{(=C^0_{\mathrm{CE}}(L,M))}{C^0_*(L, M)} \xrightarrow{d^0_*} \underset{(=C^1_{\mathrm{CE}}(L,M))}{C^1_*(L, M)} \xrightarrow{d^1_*} C^2_*(L, M) \xrightarrow{d^2_*} C^3_*(L, M).$$

First, we take $d^0_* := d^0_{CE}$.

Definition of the map $d_*^1 : C_*^1(L, M) \longrightarrow C_*^2(L, M)$.

An element $\varphi \in C_*^1(L; M)$ induces a map $\mathrm{ind}^1(\varphi) : L_{\bar{0}} \to M$ given by

$$\mathrm{ind}^1(\varphi)(x) = \varphi\big(x^{[p]}\big) - x^{p-1}\varphi(x).$$

Restricted cohomology for restricted Lie superalgebras

Definition of the map $d^1_* : C^1_*(L, M) \longrightarrow C^2_*(L, M)$.

An element $\varphi \in C^1_*(L; M)$ induces a map $\mathrm{ind}^1(\varphi) : L_{\bar{0}} \to M$ given by

$$\mathrm{ind}^1(\varphi)(x) = \varphi\left(x^{[p]}\right) - x^{p-1}\varphi(x).$$

Theorem (Evans-Fuchs)

1. *The map* $ind^1(\varphi)$ *is* $d^1_{CE}\varphi$*-compatible. Therefore,*

$$d^1_*(\varphi) := \left(d^1_{CE}\varphi, ind^1(\varphi)\right) \in C^2_*(L; M).$$

Restricted cohomology for restricted Lie superalgebras

Definition of the map $d^1_* : C^1_*(L, M) \longrightarrow C^2_*(L, M)$.

An element $\varphi \in C^1_*(L; M)$ induces a map $\mathrm{ind}^1(\varphi) : L_{\bar{0}} \to M$ given by

$$\mathrm{ind}^1(\varphi)(x) = \varphi\left(x^{[p]}\right) - x^{p-1}\varphi(x).$$

Theorem (Evans-Fuchs)

1. *The map* $ind^1(\varphi)$ *is* $d^1_{CE}\varphi$*-compatible. Therefore,*

$$d^1_*(\varphi) := \left(d^1_{CE}\varphi, ind^1(\varphi)\right) \in C^2_*(L; M).$$

2. *We have* $d^1_* \circ d^0_* = 0$.

Restricted cohomology for restricted Lie superalgebras

Definition of the map $d^1_* : C^1_*(L, M) \longrightarrow C^2_*(L, M)$.

An element $\varphi \in C^1_*(L; M)$ induces a map $\mathrm{ind}^1(\varphi) : L_{\bar{0}} \to M$ given by

$$\mathrm{ind}^1(\varphi)(x) = \varphi\left(x^{[p]}\right) - x^{p-1}\varphi(x).$$

Theorem (Evans-Fuchs)

1. *The map* $ind^1(\varphi)$ *is* $d^1_{CE}\varphi$*-compatible. Therefore,*

$$d^1_*(\varphi) := \left(d^1_{CE}\varphi, ind^1(\varphi)\right) \in C^2_*(L; M).$$

2. *We have* $d^1_* \circ d^0_* = 0$.

3. *The space* $H^1_*(L; M) := Ker\left(d^1_*\right) / Im\left(d^0_*\right)$ *is well defined.*

Restricted cohomology for restricted Lie superalgebras

Definition of the map $d^2_* : C^2_*(L, M) \longrightarrow C^3_*(L, M)$.

An element $(\varphi, \omega) \in C^2_*(L; M)$ induces a map $\mathrm{ind}^2(\varphi, \omega) : L \times L_{\bar{0}} \to M$ defined by

$$\begin{aligned}\mathrm{ind}^2(\varphi,\omega)(x,y) =& \varphi\left(x, y^{[p]}\right) - \sum_{i+j=p-1} (-1)^i y^i \varphi\Big([[\cdots[x, \overbrace{y],\cdots],y}^{j \text{ terms}}], y\Big) \\ &+ (-1)^{|\varphi||x|} x\omega(y),\end{aligned}$$

for $\mathrm{Im}(\omega) \subseteq M_{|\varphi|}$, and then extended using (1).

Restricted cohomology for restricted Lie superalgebras

Definition of the map $d^2_*: C^2_*(L, M) \longrightarrow C^3_*(L, M)$.

An element $(\varphi, \omega) \in C^2_*(L; M)$ induces a map $\operatorname{ind}^2(\varphi, \omega) : L \times L_{\bar{0}} \to M$ defined by

$$\operatorname{ind}^2(\varphi, \omega)(x, y) = \varphi\left(x, y^{[p]}\right) - \sum_{i+j=p-1} (-1)^i y^i \varphi\Big([[\cdots[x, \overbrace{y], \cdots], y}^{j \text{ terms}}], y\Big) + (-1)^{|\varphi||x|} x\omega(y),$$

for $\operatorname{Im}(\omega) \subseteq M_{|\varphi|}$, and then extended using (1).

Theorem (Bouarroudj-E.)

1. *The map $\operatorname{ind}^2(\varphi, \omega)$ is $d^2_{CE}\varphi$-compatible.*

Restricted cohomology for restricted Lie superalgebras

Definition of the map $d_*^2 : C_*^2(L, M) \longrightarrow C_*^3(L, M)$.

An element $(\varphi, \omega) \in C_*^2(L; M)$ induces a map $\mathrm{ind}^2(\varphi, \omega) : L \times L_{\bar{0}} \to M$ defined by

$$\mathrm{ind}^2(\varphi,\omega)(x,y) = \varphi\left(x, y^{[p]}\right) - \sum_{i+j=p-1} (-1)^i y^i \varphi\Big([[\cdots[x, \overbrace{y], \cdots], y}^{j \text{ terms}}], y\Big) + (-1)^{|\varphi||x|} x\omega(y),$$

for $\mathrm{Im}(\omega) \subseteq M_{|\varphi|}$, and then extended using (1).

Theorem (Bouarroudj-E.)

1. *The map* $\mathrm{ind}^2(\varphi, \omega)$ *is* $d^2_{CE}\varphi$*-compatible.*
2. *We have* $d_*^2 \circ d_*^1 = 0$, *where* $d_*^2(\varphi, \omega) := \left(d^2_{CE}\varphi, \mathrm{ind}^2(\varphi, \omega)\right)$.

Definition of the map $d_*^2 : C_*^2(L, M) \longrightarrow C_*^3(L, M)$.

An element $(\varphi, \omega) \in C_*^2(L; M)$ induces a map $\text{ind}^2(\varphi, \omega) : L \times L_{\bar{0}} \to M$ defined by

$$\begin{aligned}\text{ind}^2(\varphi,\omega)(x,y) =& \varphi\left(x, y^{[p]}\right) - \sum_{i+j=p-1} (-1)^i y^i \varphi\Big([[\cdots[x, \overbrace{y], \cdots], y}^{j \text{ terms}}], y\Big) \\ &+ (-1)^{|\varphi||x|} x\omega(y),\end{aligned}$$

for $\text{Im}(\omega) \subseteq M_{|\varphi|}$, and then extended using (1).

Theorem (Bouarroudj-E.)

1. *The map $ind^2(\varphi, \omega)$ is $d_{CE}^2\varphi$-compatible.*
2. *We have $d_*^2 \circ d_*^1 = 0$, where $d_*^2(\varphi, \omega) := \left(d_{CE}^2\varphi, ind^2(\varphi, \omega)\right)$.*
3. *The space $H_*^2(L; M) := Ker\left(d_*^2\right) / Im\left(d_*^1\right)$ is well defined.*

Example of computation

An example. Consider the Lie superalgebra

$$L = < e_1 | e_2, e_3 >, \ [e_1, e_2] = e_3, \ e_1^{[p]} = 0.$$

Let $\varphi \in C^2_{\mathrm{CE}}(L; L)$. A map $\omega : L_{\bar{0}} \to L$ is φ-compatible if and only if

$$\omega(\lambda x) = \lambda^p \omega(x) \text{ and } \omega(x + y) = \omega(x) + \omega(y), \ \forall x, y \in L_{\bar{0}}, \ \forall \lambda \in \mathbb{K}. \quad (2)$$

Example of computation

An example. Consider the Lie superalgebra

$$L = < e_1 | e_2, e_3 >, \; [e_1, e_2] = e_3, \; e_1^{[p]} = 0.$$

Let $\varphi \in C^2_{\mathrm{CE}}(L; L)$. A map $\omega : L_{\bar{0}} \to L$ is φ-compatible if and only if

$$\omega(\lambda x) = \lambda^p \omega(x) \text{ and } \omega(x+y) = \omega(x) + \omega(y), \; \forall x, y \in L_{\bar{0}}, \; \forall \lambda \in \mathbb{K}. \quad (2)$$

Lemma

A basis for the Chevalley-Eilenberg 2-cocycles space $Z^2_{CE}(L; L)$ is given by

$$\begin{aligned} \varphi_1 &= e_1 \otimes \Delta_{1,2} + 2e_2 \otimes \Delta_{2,2}; & \varphi_2 &= -2e_1 \otimes \Delta_{1,3} + 2e_3 \otimes \Delta_{3,3}; \\ \varphi_3 &= e_2 \otimes \Delta_{1,2}; & \varphi_4 &= e_2 \otimes \Delta_{1,3}; \\ \varphi_5 &= 2e_2 \otimes \Delta_{2,2} + e_3 \otimes \Delta_{2,3}; & \varphi_6 &= e_3 \otimes \Delta_{1,2}; \\ \varphi_7 &= e_3 \otimes \Delta_{1,3}; & \varphi_8 &= e_3 \otimes \Delta_{2,2}, \end{aligned}$$

where $\Delta_{i,j}(e_k, e_l) = \delta_{i,k}\delta_{j,l}$ and $\Delta_{i,j} = -(-1)^{|e_i||e_j|}\Delta_{j,i}$.

Example of computation ($p > 3$)

The case where $p > 3$. Let $(\varphi, \omega) \in C^2_*(L; L)$.

Then,

1. $(\varphi, \omega) \in Z^2_*(L; L)$ if and only if $\varphi \in Z^2_{\mathrm{CE}}(L; L)$ and $\omega(e_1) = \gamma e_3$, $\gamma \in \mathbb{K}$;

Example of computation ($p > 3$)

The case where $p > 3$. Let $(\varphi, \omega) \in C^2_*(L; L)$.

Then,

1. $(\varphi, \omega) \in Z^2_*(L; L)$ if and only if $\varphi \in Z^2_{\mathrm{CE}}(L; L)$ and $\omega(e_1) = \gamma e_3$, $\gamma \in \mathbb{K}$;
2. φ_6 and φ_8 are Chevalley-Eilenberg coboundaries;

Example of computation ($p > 3$)

The case where $p > 3$. Let $(\varphi, \omega) \in C^2_*(L; L)$.

Then,

1. $(\varphi, \omega) \in Z^2_*(L; L)$ if and only if $\varphi \in Z^2_{\mathrm{CE}}(L; L)$ and $\omega(e_1) = \gamma e_3$, $\gamma \in \mathbb{K}$;
2. φ_6 and φ_8 are Chevalley-Eilenberg coboundaries;
3. $\varphi_5 \cong \varphi_1$ and $\varphi_7 \cong \varphi_3$;

Example of computation ($p > 3$)

The case where $p > 3$. Let $(\varphi, \omega) \in C^2_*(L; L)$.

Then,

1. $(\varphi, \omega) \in Z^2_*(L; L)$ if and only if $\varphi \in Z^2_{\mathrm{CE}}(L; L)$ and $\omega(e_1) = \gamma e_3$, $\gamma \in \mathbb{K}$;
2. φ_6 and φ_8 are Chevalley-Eilenberg coboundaries;
3. $\varphi_5 \cong \varphi_1$ and $\varphi_7 \cong \varphi_3$;
4. $\mathrm{ind}^1(\psi) = 0, \ \forall \psi \in C^1_*(L; L)$.

Example of computation ($p > 3$)

The case where $p > 3$. Let $(\varphi, \omega) \in C^2_*(L; L)$.

Then,

1. $(\varphi, \omega) \in Z^2_*(L; L)$ if and only if $\varphi \in Z^2_{\mathrm{CE}}(L; L)$ and $\omega(e_1) = \gamma e_3$, $\gamma \in \mathbb{K}$;
2. φ_6 and φ_8 are Chevalley-Eilenberg coboundaries;
3. $\varphi_5 \cong \varphi_1$ and $\varphi_7 \cong \varphi_3$;
4. $\mathrm{ind}^1(\psi) = 0, \ \forall \psi \in C^1_*(L; L)$.

Therefore, we have

$$H^2_*(L; L) = \mathrm{Span}\big\{(\varphi_1, 0);\ (\varphi_2, 0);\ (\varphi_3, 0);\ (\varphi_4, 0);\ (0, \omega_5)\big\},$$

where $\omega_5(e_1) = e_3$.

Example of computation ($p = 3$)

The case where $p = 3$. Let $(\varphi, \omega) \in C^2_*(L; L)$. Suppose that

$$\omega(e_1) = \gamma_1 e_1 + \gamma_2 e_2 + \gamma_3 e_3, \ \gamma_1, \gamma_2, \gamma_3 \in \mathbb{K}\,.$$

Then,

1. $\forall \varphi \in Z^2_{\mathrm{CE}}(L; L)$, we have $\mathrm{ind}^2(\varphi, \omega)(e_1, e_1) = \gamma_2 e_3$.

Example of computation ($p = 3$)

The case where $p = 3$. Let $(\varphi, \omega) \in C^2_*(L; L)$. Suppose that

$$\omega(e_1) = \gamma_1 e_1 + \gamma_2 e_2 + \gamma_3 e_3, \ \gamma_1, \gamma_2, \gamma_3 \in \mathbb{K}\,.$$

Then,

1. $\forall \varphi \in Z^2_{\mathrm{CE}}(L; L)$, we have $\mathrm{ind}^2(\varphi, \omega)(e_1, e_1) = \gamma_2 e_3$.
2. For $i \neq 4$, we have $\mathrm{ind}^2(\varphi_i, \omega)(e_2, e_1) = \gamma_1 e_3$.
3. For $i = 4$, $\mathrm{ind}^2(\varphi_4, \omega)(e_2, e_1) = (1 - \gamma_1) e_3$.

Therefore,

$$H^2_*(L; L) = \mathrm{Span}\big\{(\varphi_1, 0);\ (\varphi_2, 0);\ (\varphi_3, 0);\ (\varphi_4, \omega_4);\ (0, \omega_5)\big\},$$

where $\omega_4(e_1) = e_1$ and $\omega_5(e_1) = e_3$.

A subcomplex

Let L be a restricted Lie superalgebra and M a restricted L-module. We define a subspace $C^2_*(L;M)^+ \subset C^2_*(L;M)$ by

$$C^2_*(L;M)^+ := \Big\{(\varphi,\omega) \in C^2_*(L;M),\ \mathrm{Im}(\omega) \subseteq M_{\bar{0}}\Big\}.$$

A subcomplex

Let L be a restricted Lie superalgebra and M a restricted L-module. We define a subspace $C^2_*(L;M)^+ \subset C^2_*(L;M)$ by

$$C^2_*(L;M)^+ := \Big\{(\varphi,\omega) \in C^2_*(L;M),\ \mathrm{Im}(\omega) \subseteq M_{\bar{0}}\Big\}.$$

Lemma

(i) *We have an inclusion* $B^2_*(L;M)_{\bar{0}} \subset C^2_*(L;M)^+$.

(ii) *The space* $C^2_*(L;M)^+$ *is* $\mathbb{Z}_2$*-graded and the degree of an homogeneous element* $(\varphi,\omega) \in C^2_*(L;M)^+$ *is given by* $|(\varphi,\omega)| = |\varphi|$.

This Lemma allows us to consider the space $Z^2_*(L;M)^+ := \ker\big(d^2_{*|C^2_*(L;M)^+}\big)$. Thus we can define

$$H^2_*(L;M)^+ := Z^2_*(L;M)^+/B^2_*(L;M)_{\bar{0}}.$$

The space $H^2_*(L;M)^+$ is $\mathbb{Z}_2$-graded.

Central extensions of restricted Lie superalgebras

Let $\left(L, [\cdot,\cdot], (\cdot)^{[p]}\right)$ be a restricted Lie superalgebra, and M be a strongly abelian restricted Lie superalgebra $\left(\textit{i.e}, [m,n] = 0 \ \forall m, n \in M, \text{ and } m^{[p]} = 0 \ \forall m \in M_{\bar{0}}\right)$.

A **restricted extension** of L by M is a short exact sequence of restricted Lie superalgebras

$$0 \longrightarrow M \xrightarrow{\iota} E \xrightarrow{\pi} L \longrightarrow 0.$$

Central extensions of restricted Lie superalgebras

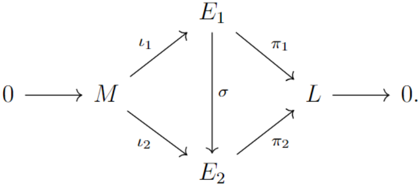
Let $\left(L, [\cdot,\cdot], (\cdot)^{[p]}\right)$ be a restricted Lie superalgebra, and M be a strongly abelian restricted Lie superalgebra $\left(i.e,\ [m,n] = 0\ \forall m,n \in M,\ \text{and } m^{[p]} = 0\ \forall m \in M_{\bar{0}}\right)$.

A **restricted extension** of L by M is a short exact sequence of restricted Lie superalgebras

$$0 \longrightarrow M \xrightarrow{\iota} E \xrightarrow{\pi} L \longrightarrow 0.$$

In the case where $\iota(M) \subset \mathfrak{z}(E) := \{a \in E,\ [a,b] = 0\ \forall b \in E\}$, M is a trivial L-module. These extensions are called **restricted central extensions**.

Two restricted central extensions of L by M are called **equivalent** if there is a restricted Lie superalgebras morphism $\sigma : E_1 \to E_2$ such that the following diagram commutes:



Central extensions of restricted Lie superalgebras

$$0 \longrightarrow M \xrightarrow{\iota} E \xrightarrow{\pi} L \longrightarrow 0.$$

Theorem (Bouarroudj-E.)

Let L be a restricted Lie superalgebra and M a strongly abelian restricted Lie superalgebra. Then, the equivalence classes of restricted central extensions of L by M are classified by $H^2_*(L; M)^+_{\bar{0}}$.

Central extensions of restricted Lie superalgebras

$$0 \longrightarrow M \stackrel{\iota}{\longrightarrow} E \stackrel{\pi}{\longrightarrow} L \longrightarrow 0.$$

Theorem (Bouarroudj-E.)

Let L be a restricted Lie superalgebra and M a strongly abelian restricted Lie superalgebra. Then, the equivalence classes of restricted central extensions of L by M are classified by $H^2_*(L; M)^+_{\bar{0}}$.

Structure maps on E. Let $(\varphi, \omega) \in Z^2_*(L; \mathbb{K})^+_{\bar{0}}$. The bracket and the p- map on E are given by

$$[x+m, y+n]_E := [x, y] + \varphi(x, y), \ \forall x, y \in L, \ \forall m, n \in M; \tag{3}$$

$$(x+m)^{[p]_E} := (x)^{[p]} + \omega(x), \ \forall x \in L_{\bar{0}}, \ \forall m \in M_{\bar{0}}. \tag{4}$$

A brief history of classification of restricted Lie algebras

Hamid Usefi

Salvatore Siciliano

- 2016 (Schneider and Usefi): Classification of p-nilpotent restricted Lie algebras of dimension ≤ 4 (Forum Math.);

A brief history of classification of restricted Lie algebras

Hamid Usefi

Salvatore Siciliano

- 2016 (Schneider and Usefi): Classification of p-nilpotent restricted Lie algebras of dimension ≤ 4 (Forum Math.);
- 2016 (Darijani and Usefi): Classification of p-nilpotent restricted Lie algebras of dimension 5, $p \geq 3$, *contains some mistakes* (J. Algebra);

A brief history of classification of restricted Lie algebras

Hamid Usefi

Salvatore Siciliano

- 2016 (Schneider and Usefi): Classification of p-nilpotent restricted Lie algebras of dimension ≤ 4 (Forum Math.);
- 2016 (Darijani and Usefi): Classification of p-nilpotent restricted Lie algebras of dimension 5, $p \geq 3$, *contains some mistakes* (J. Algebra);
- 2023 (Maletesta and Siciliano): Classification of p-nilpotent restricted Lie algebras of dimension 5, $p > 3$, *using another method* (J. Algebra).

A brief history of classification of restricted Lie algebras

Hamid Usefi

Salvatore Siciliano

- 2016 (Schneider and Usefi): Classification of p-nilpotent restricted Lie algebras of dimension ≤ 4 (Forum Math.);
- 2016 (Darijani and Usefi): Classification of p-nilpotent restricted Lie algebras of dimension 5, $p \geq 3$, *contains some mistakes* (J. Algebra);
- 2023 (Maletesta and Siciliano): Classification of p-nilpotent restricted Lie algebras of dimension 5, $p > 3$, *using another method* (J. Algebra).

Proposition

Let L be a p-nilpotent restricted Lie superalgebra of dimension n. Then, L is isomorphic to a central extension by a restricted 2-cocycle of a p-nilpotent restricted Lie superalgebra of dimension $n - 1$.

Dimension 3

- sdim$(L) = (1|2)$: $L = \langle e_1 | e_2, e_3 \rangle$.
  1. $\mathbf{L}^1_{1|2} = \langle e_1 | e_2, e_3 \rangle$ (abelian):
     1. $e_1^{[p]} = 0$;
  2. $\mathbf{L}^2_{1|2} = \langle e_1 | e_2, e_3; [e_2, e_3] = e_1 \rangle$:
     1. $e_1^{[p]} = 0$;
  3. $\mathbf{L}^3_{1|2} = \langle e_1 | e_2, e_3; [e_1, e_2] = e_3 \rangle$:
     1. $e_1^{[p]} = 0$.
  4. $\mathbf{L}^4_{1|2} = \langle e_1 | e_2, e_3; [e_3, e_3] = e_1 \rangle$:
     1. $e_1^{[p]} = 0$;
- sdim$(L) = (2|1)$: $L = \langle e_1, e_2 | e_3 \rangle$.
  1. $\mathbf{L}^1_{2|1} = \langle e_1, e_2 | e_3 \rangle$ (abelian):
     1. $e_1^{[p]} = e_2^{[p]} = 0$;
     2. $e_1^{[p]} = e_2, \; e_2^{[p]} = 0$.
  2. $\mathbf{L}^2_{2|1} = \langle e_1, e_2 | e_3; [e_3, e_3] = e_2 \rangle$:
     1. $e_1^{[p]} = e_2^{[p]} = 0$;
     2. $e_1^{[p]} = e_2, \; e_2^{[p]} = 0$.
- sdim$(L) = (3|0)$: $L = \langle e_1, e_2, e_3 \rangle$, (see Schneider-Usefi).
  1. $\mathbf{L}^1_{3|0} = \langle e_1, e_2, e_3 \rangle$ (abelian):
     1. $e_1^{[p]} = e_2^{[p]} = e_3^{[p]} = 0$;
     2. $e_1^{[p]} = e_2, \; e_2^{[p]} = e_3^{[p]} = 0$;
     3. $e_1^{[p]} = e_2, \; e_2^{[p]} = e_3, \; e_3^{[p]} = 0$.
  2. $\mathbf{L}^2_{3|0} = \langle e_1, e_2, e_3; [e_1, e_2] = e_3 \rangle$
     1. $e_1^{[p]} = e_2^{[p]} = e_3^{[p]} = 0$;
     2. $e_1^{[p]} = e_3, \; e_2^{[p]} = e_3^{[p]} = 0$.

The classification method

1. For each 3-dimensional Lie superalgebra of the previous list, we compute the equivalence classes of non-trivial *ordinary* 2-cocycles under the action by automorphisms given by

$$(A\varphi)(x,y) = \varphi\big(A(x), A(y)\big), \ \forall x, y \in L \tag{5}$$

2. We build the corresponding central extensions.
3. Some of the superalgebras obtained are isomorphic. We detect and remove redundancies.
4. Using Jacobson's Theorem, we check whether the p-maps on the even part are compatible with the odd part.

Dimension 4: scalar restricted 2-cocycles

Notation: Let $L = L_{\bar{0}} \oplus L_{\bar{1}} = \langle e_1, \cdots, e_n | e_{n+1}, \cdots, e_{n+m} \rangle$ be a restricted Lie superalgebra of superdimension $\mathrm{sdim}(L) = (n|m)$. A basis for (ordinary) 2-cocycles is then given by

$$\Delta_{i,j} : L \times L \longrightarrow \mathbb{K}, \qquad 1 \leq i \leq n+m, \; i \leq j \leq n+m,$$

where $\Delta_{i,j}(e_k, e_l) = \delta_{i,k}\delta_{j,l}$ and $\Delta_{i,j} = -(-1)^{|e_i||e_j|}\Delta_{j,i}$.

Dimension 4: scalar restricted 2-cocycles

Notation: Let $L = L_{\bar{0}} \oplus L_{\bar{1}} = \langle e_1, \cdots, e_n | e_{n+1}, \cdots, e_{n+m} \rangle$ be a restricted Lie superalgebra of superdimension $\mathrm{sdim}(L) = (n|m)$. A basis for (ordinary) 2-cocycles is then given by

$$\Delta_{i,j} : L \times L \longrightarrow \mathbb{K}, \qquad 1 \leq i \leq n+m, \; i \leq j \leq n+m,$$

where $\Delta_{i,j}(e_k, e_l) = \delta_{i,k}\delta_{j,l}$ and $\Delta_{i,j} = -(-1)^{|e_i||e_j|}\Delta_{j,i}$.

Theorem

Suppose that L is a nilpotent Lie superalgebra of total dimension 3 *with* $\dim(L_{\bar{1}}) \geq 1$ *over an algebraically closed field of characteristic* $p \geq 3$*. The equivalence classes of (ordinary) non trivial homogeneous* 2*-cocycles on L are given by*

$L = \mathbf{L}^{1}_{0|3}$: $\Delta_{1,1}, \;\; \Delta_{1,2}, \; \Delta_{1,1} + \Delta_{2,3}$;

$L = \mathbf{L}^{1}_{1|2}$: $\Delta_{1,2}, \; \Delta_{2,3}, \; \Delta_{2,2} + \Delta_{2,3} + \Delta_{3,3}$;

$L = \mathbf{L}^{2}_{1|2}$: $\Delta_{2,2}, \; \Delta_{2,2} + \Delta_{3,3}$;

$L = \mathbf{L}^{3}_{1|2}$: $\Delta_{1,3}, \; \Delta_{2,2}$;

$L = \mathbf{L}^{4}_{1|2}$: $\Delta_{2,2}, \; \Delta_{2,3}, \; \Delta_{2,2} + \Delta_{2,3}$.

$L = \mathbf{L}^{1}_{2|1}$: $\Delta_{1,3}, \; \Delta_{1,2}, \; \Delta_{3,3}, \; \Delta_{1,2} + \Delta_{3,3}$;

$L = \mathbf{L}^{2}_{2|1}$: $\Delta_{1,3}$.

Dimension 4: the classification. Building the extensions.

With the list of 2-cocycles, we can extend the Lie brackets using

$$[x, y]_{\text{new}} = [x, y]_{\text{old}} + \Delta(x, y)X. \tag{6}$$

Dimension 4: the classification. Building the extensions.

With the list of 2-cocycles, we can extend the Lie brackets using

$$[x, y]_{\text{new}} = [x, y]_{\text{old}} + \Delta(x, y)X. \quad (6)$$

Example. Consider $\mathbf{L}^{\mathbf{3}}_{\mathbf{1|2}} = \langle e_1 | e_2, e_3; [e_1, e_2] = e_3 \rangle$. The 2-cocycles are $\Delta_{1,3}$ and $\Delta_{2,2}$. We obtain four superalgebras of dimension 4.

Name	sdim	Cocycle	Added element	Bracket
$\mathbf{L}^{\mathbf{g}}_{\mathbf{2|2}}$	$(2|2)$	0	X even	$[e_1, e_2] = e_3$
$\mathbf{L}^{\mathbf{d}}_{\mathbf{1|3}}$	$(1|3)$	0	X odd	$[e_1, e_2] = e_3$
$\mathbf{L}^{\mathbf{e}}_{\mathbf{1|3}}$	$(1|3)$	$\Delta_{1,3}$	X odd	$[e_1, e_2] = e_3, \; [e_1, e_3] = X$
$\mathbf{L}^{\mathbf{h}}_{\mathbf{2|2}}$	$(2|2)$	$\Delta_{2,2}$	X even	$[e_1, e_2] = e_3, \; [e_2, e_2] = X$

Lie superalgebras obtained by central extensions of $\mathbf{L}^{\mathbf{3}}_{\mathbf{1|2}}$.

Dimension 4: the classification. Building the extensions.

Name	sdim	Cocycle	Added element	Bracket
$\mathbf{L}^{\mathbf{g}}_{\mathbf{2\vert 2}}$	$(2\vert 2)$	0	X even	$[e_1, e_2] = e_3$
$\mathbf{L}^{\mathbf{d}}_{\mathbf{1\vert 3}}$	$(1\vert 3)$	0	X odd	$[e_1, e_2] = e_3$
$\mathbf{L}^{\mathbf{e}}_{\mathbf{1\vert 3}}$	$(1\vert 3)$	$\Delta_{1,3}$	X odd	$[e_1, e_2] = e_3,\ [e_1, e_3] = X$
$\mathbf{L}^{\mathbf{h}}_{\mathbf{2\vert 2}}$	$(2\vert 2)$	$\Delta_{2,2}$	X even	$[e_1, e_2] = e_3,\ [e_2, e_2] = X$

Lie superalgebras obtained by central extensions of $\mathbf{L}^{\mathbf{3}}_{\mathbf{1\vert 2}}$.

Name	sdim	Cocycle	Added element	Bracket
$\mathbf{L}^{\mathbf{a}}_{\mathbf{2\vert 2}}$	$(2\vert 2)$	0	X even	$[\cdot, \cdot] = 0$
$\mathbf{L}^{\mathbf{a}}_{\mathbf{1\vert 3}}$	$(1\vert 3)$	0	X odd	$[\cdot, \cdot] = 0$
$\mathbf{L}^{\mathbf{b}}_{\mathbf{1\vert 3}}$	$(1\vert 3)$	$\Delta_{1,2}$	X odd	$[e_1, e_3] = X$
$\mathbf{L}^{\mathbf{b}}_{\mathbf{2\vert 2}}$	$(2\vert 2)$	$\Delta_{2,3}$	X even	$[e_2, e_3] = X$
$\mathbf{L}^{\mathbf{c}}_{\mathbf{2\vert 2}}$	$(2\vert 2)$	$\Delta_{2,2} + \Delta_{2,3} + \Delta_{3,3}$	X even	$[e_2, e_2] = [e_2, e_3] = [e_3, e_3] = X$

Lie superalgebras obtained by extensions of $\mathbf{L}^{\mathbf{1}}_{\mathbf{1\vert 2}}$ (abelian).

Dimension 4: the classification. Building the extensions.

Name	sdim	Cocycle	Added element	Bracket
$\mathbf{L}^{\mathbf{g}}_{\mathbf{2}\mid\mathbf{2}}$	$(2\mid 2)$	0	X even	$[e_1, e_2] = e_3$
$\mathbf{L}^{\mathbf{d}}_{\mathbf{1}\mid\mathbf{3}}$	$(1\mid 3)$	0	X odd	$[e_1, e_2] = e_3$
$\mathbf{L}^{\mathbf{e}}_{\mathbf{1}\mid\mathbf{3}}$	$(1\mid 3)$	$\Delta_{1,3}$	X odd	$[e_1, e_2] = e_3,\ [e_1, e_3] = X$
$\mathbf{L}^{\mathbf{h}}_{\mathbf{2}\mid\mathbf{2}}$	$(2\mid 2)$	$\Delta_{2,2}$	X even	$[e_1, e_2] = e_3,\ [e_2, e_2] = X$

Lie superalgebras obtained by central extensions of $\mathbf{L}^{\mathbf{3}}_{\mathbf{1}\mid\mathbf{2}}$.

Name	sdim	Cocycle	Added element	Bracket
$\mathbf{L}^{\mathbf{a}}_{\mathbf{2}\mid\mathbf{2}}$	$(2\mid 2)$	0	X even	$[\cdot,\cdot] = 0$
$\mathbf{L}^{\mathbf{a}}_{\mathbf{1}\mid\mathbf{3}}$	$(1\mid 3)$	0	X odd	$[\cdot,\cdot] = 0$
$\mathbf{L}^{\mathbf{b}}_{\mathbf{1}\mid\mathbf{3}}$	$(1\mid 3)$	$\Delta_{1,2}$	X odd	$[e_1, e_3] = X$
$\mathbf{L}^{\mathbf{b}}_{\mathbf{2}\mid\mathbf{2}}$	$(2\mid 2)$	$\Delta_{2,3}$	X even	$[e_2, e_3] = X$
$\mathbf{L}^{\mathbf{c}}_{\mathbf{2}\mid\mathbf{2}}$	$(2\mid 2)$	$\Delta_{2,2} + \Delta_{2,3} + \Delta_{3,3}$	X even	$[e_2, e_2] = [e_2, e_3] = [e_3, e_3] = X$

Lie superalgebras obtained by extensions of $\mathbf{L}^{\mathbf{1}}_{\mathbf{1}\mid\mathbf{2}}$ (abelian).

We know that $\mathbf{L}^{\mathbf{d}}_{\mathbf{1}\mid\mathbf{3}} \ncong \mathbf{L}^{\mathbf{e}}_{\mathbf{1}\mid\mathbf{3}}$ and $\mathbf{L}^{\mathbf{a}}_{\mathbf{1}\mid\mathbf{3}} \ncong \mathbf{L}^{\mathbf{b}}_{\mathbf{1}\mid\mathbf{3}}$...

Dimension 4: the classification. Building the extensions.

Name	sdim	Cocycle	Added element	Bracket
$\mathbf{L}^{\mathbf{g}}_{\mathbf{2}\mid\mathbf{2}}$	$(2\mid 2)$	0	X even	$[e_1, e_2] = e_3$
$\mathbf{L}^{\mathbf{d}}_{\mathbf{1}\mid\mathbf{3}}$	$(1\mid 3)$	0	X odd	$[e_1, e_2] = e_3$
$\mathbf{L}^{\mathbf{e}}_{\mathbf{1}\mid\mathbf{3}}$	$(1\mid 3)$	$\Delta_{1,3}$	X odd	$[e_1, e_2] = e_3, \; [e_1, e_3] = X$
$\mathbf{L}^{\mathbf{h}}_{\mathbf{2}\mid\mathbf{2}}$	$(2\mid 2)$	$\Delta_{2,2}$	X even	$[e_1, e_2] = e_3, \; [e_2, e_2] = X$

Lie superalgebras obtained by central extensions of $\mathbf{L}^{\mathbf{3}}_{\mathbf{1}\mid\mathbf{2}}$.

Name	sdim	Cocycle	Added element	Bracket
$\mathbf{L}^{\mathbf{a}}_{\mathbf{2}\mid\mathbf{2}}$	$(2\mid 2)$	0	X even	$[\cdot, \cdot] = 0$
$\mathbf{L}^{\mathbf{a}}_{\mathbf{1}\mid\mathbf{3}}$	$(1\mid 3)$	0	X odd	$[\cdot, \cdot] = 0$
$\mathbf{L}^{\mathbf{b}}_{\mathbf{1}\mid\mathbf{3}}$	$(1\mid 3)$	$\Delta_{1,2}$	X odd	$[e_1, e_3] = X$
$\mathbf{L}^{\mathbf{b}}_{\mathbf{2}\mid\mathbf{2}}$	$(2\mid 2)$	$\Delta_{2,3}$	X even	$[e_2, e_3] = X$
$\mathbf{L}^{\mathbf{c}}_{\mathbf{2}\mid\mathbf{2}}$	$(2\mid 2)$	$\Delta_{2,2} + \Delta_{2,3} + \Delta_{3,3}$	X even	$[e_2, e_2] = [e_2, e_3] = [e_3, e_3] = X$

Lie superalgebras obtained by extensions of $\mathbf{L}^{\mathbf{1}}_{\mathbf{1}\mid\mathbf{2}}$ (abelian).

We know that $\mathbf{L}^{\mathbf{d}}_{\mathbf{1}\mid\mathbf{3}} \not\cong \mathbf{L}^{\mathbf{e}}_{\mathbf{1}\mid\mathbf{3}}$ and $\mathbf{L}^{\mathbf{a}}_{\mathbf{1}\mid\mathbf{3}} \not\cong \mathbf{L}^{\mathbf{b}}_{\mathbf{1}\mid\mathbf{3}}$...

But $\mathbf{L}^{\mathbf{b}}_{\mathbf{1}\mid\mathbf{3}} \cong \mathbf{L}^{\mathbf{d}}_{\mathbf{1}\mid\mathbf{3}}$.

Dimension 4: the classification. Detecting isomorphisms.

It is possible that two superalgebras obtained as central extensions by non-equivalent cocycles are isomorphic. We need to detect and remove redundancies.

L	$[L, L]$	$\mathrm{sdim}(\mathfrak{z}(L))$	$\mathrm{sdim}\left(H^1_{\mathrm{CE}}(L;\mathbb{K})\right)$	$\mathrm{sdim}\left(H^2_{\mathrm{CE}}(L;\mathbb{K})\right)$	$\mathrm{sdim}\left(H^3_{\mathrm{CE}}(L;\mathbb{K})\right)$
$\mathbf{L^a_{1\|3}}$	0	$1\|3$	$1\|3$	$6\|3$	$7\|9$
$\mathbf{L^b_{1\|3}}$	$\langle X\rangle$	$0\|2$	$1\|2$	$3\|2$	$3\|4$ ($3\|5$ if $p=3$)
$\mathbf{L^c_{1\|3}}$	$\langle e_1\rangle$	$1\|1$	$0\|3$	$5\|0$	$0\|7$
$\mathbf{L^e_{1\|3}}$	$\langle e_3, X\rangle$	$0\|1$	$1\|1$	$2\|1$	$2\|2$ ($2\|4$ if $p=3$)
$\mathbf{L^f_{1\|3}}$	$\langle e_1\rangle$	$1\|2$	$0\|3$	$5\|0$	$0\|7$
$\mathbf{L^j_{1\|3}}$	$\langle X\rangle$	$1\|0$	$0\|3$	$5\|0$	$0\|7$

Invariants for Lie superalgebras of sdim $= (1|3)$.

Dimension 4: the classification. Lie superalgebras.

Theorem

The classification of 4-dimensional nilpotent Lie superalgebras over an algebraically closed field of characteristic different from 2 is given by:

$\underline{sdim(L) = (0|4)}$: $L = \langle 0|x_1, x_2, x_3, x_4\rangle$

$\mathbf{L}^{\mathbf{1}}_{\mathbf{0|4}}$: $[\cdot,\cdot] = 0$.

$\underline{sdim(L) = (1|3)}$: $L = \langle x_1|x_2, x_3, x_4\rangle$

$\mathbf{L}^{\mathbf{1}}_{\mathbf{1|3}}$ $(= \mathbf{L}^{\mathbf{a}}_{\mathbf{1|3}})$: *abelian;*

$\mathbf{L}^{\mathbf{2}}_{\mathbf{1|3}}$ $(= \mathbf{L}^{\mathbf{b}}_{\mathbf{1|3}})$: $[x_1, x_3] = x_4$;

$\mathbf{L}^{\mathbf{3}}_{\mathbf{1|3}}$ $(= \mathbf{L}^{\mathbf{c}}_{\mathbf{1|3}})$: $[x_2, x_3] = x_1$;

$\mathbf{L}^{\mathbf{4}}_{\mathbf{1|3}}$ $(= \mathbf{L}^{\mathbf{e}}_{\mathbf{1|3}})$: $[x_1, x_2] = x_3$, $[x_1, x_3] = x_4$;

$\mathbf{L}^{\mathbf{5}}_{\mathbf{1|3}}$ $(= \mathbf{L}^{\mathbf{f}}_{\mathbf{1|3}})$: $[x_3, x_3] = x_1$;

$\mathbf{L}^{\mathbf{6}}_{\mathbf{1|3}}$ $(= \mathbf{L}^{\mathbf{j}}_{\mathbf{1|3}})$: $[x_2, x_2] = x_1$, $[x_3, x_4] = x_1$.

$\underline{sdim(L) = (2|2)}$: $L = \langle x_1, x_2|x_3, x_4\rangle$

$\mathbf{L}^{\mathbf{1}}_{\mathbf{2|2}}$ $(= \mathbf{L}^{\mathbf{a}}_{\mathbf{2|2}})$: *abelian;*

$\mathbf{L}^{\mathbf{2}}_{\mathbf{2|2}}$ $(= \mathbf{L}^{\mathbf{b}}_{\mathbf{2|2}})$: $[x_3, x_4] = x_2$;

$\mathbf{L}^{\mathbf{3}}_{\mathbf{2|2}}$ $(= \mathbf{L}^{\mathbf{e}}_{\mathbf{2|2}})$: $[x_3, x_3] = x_2$, $[x_3, x_4] = x_1$;

$\mathbf{L}^{\mathbf{4}}_{\mathbf{2|2}}$ $(= \mathbf{L}^{\mathbf{f}}_{\mathbf{2|2}})$: $[x_3, x_3] = [x_4, x_4] = x_2$, $[x_3, x_4] = x_1$;

$\mathbf{L}^{\mathbf{5}}_{\mathbf{2|2}}$ $(= \mathbf{L}^{\mathbf{g}}_{\mathbf{2|2}})$: $[x_1, x_3] = x_4$;

$\mathbf{L}^{\mathbf{6}}_{\mathbf{2|2}}$ $(= \mathbf{L}^{\mathbf{h}}_{\mathbf{2|2}})$: $[x_1, x_3] = x_4$, $[x_3, x_3] = x_2$.

$\mathbf{L}^{\mathbf{7}}_{\mathbf{2|2}}$ $(= \mathbf{L}^{\mathbf{i}}_{\mathbf{2|2}})$: $[x_4, x_4] = x_1$.

$\underline{sdim(L) = (3|1)}$: $L = \langle x_1, x_2, x_3|x_4\rangle$

$\mathbf{L}^{\mathbf{1}}_{\mathbf{3|1}}$ $(= \mathbf{L}^{\mathbf{a}}_{\mathbf{3|1}})$: *abelian;*

$\mathbf{L}^{\mathbf{2}}_{\mathbf{3|1}}$ $(= \mathbf{L}^{\mathbf{b}}_{\mathbf{3|1}})$: $[x_1, x_2] = x_3$;

$\mathbf{L}^{\mathbf{3}}_{\mathbf{3|1}}$ $(= \mathbf{L}^{\mathbf{c}}_{\mathbf{3|1}})$: $[x_2, x_2] = x_3$;

$\mathbf{L}^{\mathbf{4}}_{\mathbf{3|1}}$ $(= \mathbf{L}^{\mathbf{d}}_{\mathbf{3|1}})$: $[x_1, x_2] = [x_3, x_4] = x_3$.

$\underline{sdim(L) = (4|0)}$: $L = \langle x_1, x_2, x_3, x_4|0\rangle$

$\mathbf{L}^{\mathbf{1}}_{\mathbf{4|0}}$: *abelian;*

$\mathbf{L}^{\mathbf{2}}_{\mathbf{4|0}}$: $[x_1, x_2] = x_3$;

$\mathbf{L}^{\mathbf{3}}_{\mathbf{4|0}}$: $[x_1, x_2] = x_3$, $[x_1, x_3] = x_4$.

Dimension 4: the classification. $p|2p$ maps.

- Suppose that $L = L_{\bar{0}} \oplus L_{\bar{1}}$ is a p-nilpotent restricted Lie superalgebra. Then $L_{\bar{0}}$ is a p-nilpotent restricted Lie algebra with a p-map $(\cdot)^{[p]}$.

Dimension 4: the classification. $p|2p$ maps.

- Suppose that $L = L_{\bar{0}} \oplus L_{\bar{1}}$ is a p-nilpotent restricted Lie superalgebra. Then $L_{\bar{0}}$ is a p-nilpotent restricted Lie algebra with a p-map $(\cdot)^{[p]}$.

- The classification of 4-dimensional restricted Lie algebras has been achieved by Schneider-Usefi.

Dimension 4: the classification. $p|2p$ maps.

- Suppose that $L = L_{\bar{0}} \oplus L_{\bar{1}}$ is a p-nilpotent restricted Lie superalgebra. Then $L_{\bar{0}}$ is a p-nilpotent restricted Lie algebra with a p-map $(\cdot)^{[p]}$.

- The classification of 4-dimensional restricted Lie algebras has been achieved by Schneider-Usefi.

- We only have to check whether these p-maps satisfy

$$\mathrm{ad}_{e_i}^{p}(f_j) = \mathrm{ad}_{e_i^{[p]}}(f_j),$$

$\forall e_i$ basis elements of $L_{\bar{0}}$, $\forall f_i$ basis elements of $L_{\bar{1}}$.

Dimension 4: the classification. $p|2p$ maps.

Theorem

The p-nilpotent structures on nilpotent Lie superalgebras of total dimension 4 *with* $\dim(L_{\bar{1}}) > 0$ *are given by:*

- $sdim(L) = (0|4)$*: none.*
- $sdim(L) = (1|3)$*:* $x_1^{[p]} = 0$.
- $sdim(L) = (2|2)$*:*
  - $x_1^{[p]_1} = x_2^{[p]_1} = 0$;
  - $x_1^{[p]_2} = x_2, \ x_2^{[p]_2} = 0$.
- $sdim(L) = (3|1)$*:*
  - *Case* $L_{\bar{0}}$ *abelian:*
    - $x_1^{[p]_1} = x_2^{[p]_1} = x_3^{[p]_1} = 0$;
    - $x_1^{[p]_2} = x_2, \ x_2^{[p]_2} = x_3^{[p]_2} = 0$.
    - $x_1^{[p]_3} = x_2, \ x_2^{[p]_3} = x_3, \ x_3^{[p]_3} = 0$.
  - *Case* $L_{\bar{0}} \cong \mathbf{L}^2_{3|0} = \langle x_1, x_2, x_3; [x_1, x_2] = x_3 \rangle$*:*
    - $x_1^{[p]_4} = x_2^{[p]_4} = x_3^{[p]_4} = 0$;
    - $x_1^{[p]_5} = x_3, \ x_2^{[p]_5} = x_3^{[p]_5} = 0$.

Merci pour votre attention!