Restricted Poisson algebras in characteristic 2

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Workshop on algebras and applications in mathematical physics

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Joint work with
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Outline of the talk

1. Restricted Poisson and Lie-Rinehart algebras
2. Restricted cohomology for $p = 2$
3. A comparison between the cohomologies

Restricted Lie algebras

Definition (Jacobson)

A **restricted Lie algebra** *is a Lie algebra L equipped with a map $(-)^{[p]} : L \longrightarrow L$ satisfying for all $x, y \in L$ and for all $\lambda \in \mathbb{K}$:*

1. $(\lambda x)^{[p]} = \lambda^p x^{[p]}$;
2. $\left[x, y^{[p]}\right] = [[\cdots[x, \overbrace{y], y], \cdots, y}^{p \text{ terms}}]$;
3. $(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x, y)$,

Nathan Jacobson (1910-1999)

with is$_i(x, y)$ the coefficient of Z^{i-1} in $\mathrm{ad}_{Zx+y}^{p-1}(x)$. Such a map $(-)^{[p]} : L \longrightarrow L$ is called p-map.

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with is$_i(x,y)$ *the coefficient of* Z^{i-1} *in* $\mathrm{ad}^{p-1}_{Zx+y}(x)$*. Such a map* $(-)^{[p]} : L \longrightarrow L$ *is called* *p-map.*

Remark: in the case where $p = 2$, Condition (3) reduces to

$$(x+y)^{[2]} = x^{[2]} + y^{[2]} + [x, y].$$

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Example: any associative algebra A with $[a, b] = ab - ba$ and $a^{[p]} = a^p$, $\forall a, b \in A$.

Restricted Poisson algebras

Definition (Poisson algebra)

An associative commutative $\mathbb{K}$-algebra $(A, \cdot)$ (not necessarily unital) is called **Poisson algebra** if it is equipped with a bilinear map $\{-,-\} : A \times A \to A$ such that $\big(A, \{-,-\}\big)$ is a Lie algebra and moreover, we have

$$\{x \cdot y, z\} = x \cdot \{y, z\} + y \cdot \{x, z\}, \quad \forall x, y, z \in A. \tag{1}$$

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Question: in the case where A is restricted, is there a compatibility between the p-map and the associative product of A?

Restricted Poisson algebras

Bezrukavnikov and Kaledin, 2008 ($p \geq 3$):

$$(xy)^{\{p\}} = x^p y^{\{p\}} + y^p x^{\{p\}} + \Phi_p(x, y), \ \forall x, y \in A, \tag{2}$$

where

$$\Phi_p(x, y) = (x^p + y^p) \sum_{1 \leq i \leq p-1} s_i(x, y) - \frac{1}{2} \sum_{1 \leq i \leq p-1} s_i(x^2, y^2) + \sum_{1 \leq i \leq p-1} s_i(x^2 + y^2, 2xy).$$

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Bao, Ye and Zhang, 2017 ($p \geq 3$): Equation (2) is equivalent to

$$(x^2)^{\{p\}} = 2x^p x^{\{p\}}, \quad \forall x \in A. \qquad (3)$$

Restricted Poisson algebras

Definition (Restricted Poisson algebra)

A Poisson algebra $(A, \cdot, \{-,-\})$ over a field of characteristic $p = 2$ is called **restricted Poisson algebra** if

- $(A, \{-,-\}, (-)^{\{2\}})$ is a restricted Lie algebra;
- we have

$$(xy)^{\{2\}} = x^2 y^{\{2\}} + y^2 x^{\{2\}} + xy\{x, y\}, \ \forall x, y \in A. \tag{4}$$

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- This definition appeared already in Petrogradsky-Shestakov, Journal of Algebra 574 (2021).

- **Example**: the 3-dimensional restricted Lie algebra L spanned by e_1, e_2, e_3 with
  - the bracket $[e_2, e_3] = e_2$;
  - the 2-map $e_3^{[2]} = e_3$;
  - the associative commutative product $e_1e_1 = e_1$.

Restricted Lie-Rinehart algebras

Definition (Dokas)

A **restricted Lie-Rinehart algebra** in characteristic 2 is a triple (A, L, θ), where

- A is an associative commutative algebra;
- $\left(L, [-,-], (-)^{[2]}\right)$ is a restricted Lie algebra that is also an A-module;
- $\theta : L \to \mathrm{Der}(A)$ is an A-linear restricted Lie algebras morphism satisfying for all $x, y \in L$ and for all $a \in A$:

$$[x, ay] = a[x, y] + \theta(x)(a)y; \quad \text{and} \tag{5}$$

$$(ax)^{[2]} = a^2 x^{[2]} + \theta(ax)(a)x. \tag{6}$$

Example: A associative commutative algebra. Take $L = \mathrm{Der}(A)$ and $\theta = \mathrm{id}$.

Some constructions

Proposition

Let (A, L, θ) be a restricted Lie-Rinehart algebra. Then, the tuple $\left(A \oplus L, \cdot, \{-,-\}, (-)^{\{2\}}\right)$ is a restricted Poisson algebra, where

$$
\begin{aligned}
(a+x)\cdot(b+y) &:= ab + ay + bx; && \forall a, b \in A,\ \forall x, y \in L;\\
\{a+x, b+y\} &:= [x,y]_L + \theta(x)(b) + \theta(y)(a), && \forall a, b \in A,\ \forall x, y \in L; \qquad (7)\\
(a+x)^{\{2\}} &:= x^{[2]_L} + \theta(x)(a), && \forall a \in A,\ \forall x \in L.
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Proposition

Let (A, μ) be an associative algebra and let $\mu_t = \mu + \sum_{i\geq 1} t^i \mu_i$ be a formal deformation of μ. Then :

- *$\{a, b\} := \mu_1(a, b) + \mu_1(b, a)$ is an ordinary Poisson bracket on A;*
- *if $\mu_1(a^2, b) = 0$ and $\mu_2(a^2, b) = \mu_2(b, a^2)$ $\forall a, b \in A$, then $(A, \{-,-\}, \omega_{\mu_1})$ is a restricted Poisson algebra, where $\omega_{\mu_1}(a) = \mu_1(a, a)$.*

Restricted cohomology, $p = 2$

Let $\big(L, [-,-], (-)^{[2]}\big)$ be a restricted Lie algebra and let M be a restricted L-module. We set

$$C^0_{\mathsf{res}}(L;M) := C^0_{\mathrm{CE}}(L;M);$$
$$C^1_{\mathsf{res}}(L;M) := C^1_{\mathrm{CE}}(L;M).$$

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Let $n \geq 2$, $\boxed{\varphi \in C^n_{\text{CE}}(L, M),\ \omega : L \otimes \wedge^{n-2} L \to M,}$ $\lambda \in \mathbb{K}$ and $x, z_2, \cdots, z_{n-1} \in L$. The pair (φ, ω) is a n-cochain of the restricted cohomology if

$$\omega(\lambda x, z_2, \cdots, z_{n-1}) = \lambda^2 \omega(x, z_2, \cdots, z_{n-1}) \tag{8}$$

$$\begin{aligned}\omega(x + y, z_2, \cdots, z_{n-1}) = \omega(x, z_2, \cdots, z_{n-1}) + \omega(y, z_2, \cdots, z_{n-1})\\ + \varphi(x, y, z_2, \cdots, z_{n-1}),\end{aligned} \tag{9}$$

$$(z_2, \cdots, z_{n-1}) \mapsto \omega(-, z_2, \cdots, z_{n-1}) \text{ is linear.} \tag{10}$$

We denote by $C^n_{\text{res}}(L, M)$ the space of n-cochains of L with values in M.

Restricted cohomology, $p = 2$

For $n \geq 2$, the coboundary maps $d^n_{\text{res}} : C^n_{\text{res}}(L; M) \to C^{n+1}_{\text{res}}(L; M)$ are given by

$$d^n_{\text{res}}(\varphi, \omega) = \big(d^n_{\mathrm{CE}}(\varphi), \delta^n(\omega)\big),$$

$$\begin{aligned}
\delta^n\omega(x, z_2, \cdots, z_n) := {} & x \cdot \varphi(x, z_2, \cdots, z_n) + \sum_{i=2}^{n} z_i \cdot \omega(x, z_2, \cdots, \hat{z}_i, \cdots, z_n) \\
& + \varphi(x^{[2]}, z_2, \cdots, z_n) + \sum_{i=2}^{n} \varphi\left([x, z_i], x, z_2, \cdots, \hat{z}_i, \cdots, z_n\right) \\
& + \sum_{2 \leq i < j \leq n} \omega\left(x, [z_i, z_j], z_2, \cdots, \hat{z}_i, \cdots, \hat{z}_j, \cdots, z_n\right).
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For $n = 0, 1$, we define $d^0_{\text{res}} = d^0_{\mathrm{CE}}$ and

$$\begin{aligned} d^1_{\text{res}} : C^1_{\text{res}}(L; M) & \to C^2_{\text{res}}(L, M) \\ \varphi & \mapsto \big(d^1_{\mathrm{CE}}\varphi, \delta^1\varphi\big), \text{ where } \delta^1\varphi(x) := \varphi\big(x^{[2]}\big) + x \cdot \varphi(x), \ \forall x \in L.\end{aligned}$$

Restricted cohomology, $p = 2$

For $n = 2$, the cocycle condition on the second component reads

$$x \cdot \varphi(x, z) + z \cdot \omega(x) + \varphi(x^{[2]}, z) + \varphi([x, z], x) = 0, \quad \forall x, z \in L.$$

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Theorem

1. *Let* $(\varphi, \omega) \in C^n_{res}(L; M)$*. Then* $\big(d^n_{CE}(\varphi), \delta^n(\omega)\big) \in C^{n+1}_{res}(L; M)$*;*
2. $\delta^{n+1} \circ \delta^n = 0$*;*
3. $H^n_{res}(L; M) := Z^n_{res}(L; M)/B^n_{res}(L; M)$ *are well-defined, with*
   - $Z^n_{res}(L; M) = Ker(d^n_{res})$ *the restricted n-cocycles;*
   - $B^n_{res}(L; M) = Im(d^{n-1}_{res})$ *the restricted n-coboundaries.*

Restricted cohomology for Lie-Rinehart algebras, $p = 2$

Let (A, L, θ) be a restricted Lie-Rinehart algebra in characteristic 2 and let M be a restricted Lie-Rinehart module:

$$x(am) = (ax)m + \theta_L(x)(a)m, \ \forall x \in L, \ \forall a \in A, \ \forall m \in M.$$

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For $n = 0, 1$, we set $C^0_{\mathrm{LR}}(L; M) := M$ and $C^1_{\mathrm{LR}}(L; M) := \mathrm{Hom}_A(\wedge^1 L, M)$.

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For $n \geq 2$, $C^n_{\mathrm{LR}}(L; M)$ consists of pairs $(\varphi, \omega) \in C^n_{\mathrm{res}}(L; M)$, where $\varphi \in \mathrm{Hom}_A(\wedge^n L, M)$ and

$$\omega(ax, z_2, \cdots, z_{n-1}) = a^2\omega(x, z_2, \cdots, z_{n-1}), \quad (11)$$

$$\omega(x, z_2, \cdots, az_i, \cdots, z_{n-1}) = a\omega(x, z_2, \cdots, z_i, \cdots, z_{n-1}), \quad (12)$$

for all $x, z_2, \cdots, z_{n-1} \in L$ and for all $a \in A$.

Proposition (Bouarroudj-E.-Liu)

Define $\boxed{d^n_{\mathrm{LR}} = d^n_{\mathrm{res}|C^n_{\mathrm{LR}}(L;M)}.}$ *Then,* $d^n_{\mathrm{LR}}\left(C^n_{\mathrm{LR}}(L; M)\right) \in C^{n+1}_{\mathrm{LR}}(L; M)$ *and the spaces* $H^n_{\mathrm{LR}}(L; M)$ *are well-defined.*

Application: abelian extensions

Let (A, L, θ_L) be a restricted Lie-Rinehart algebra and let (A, M, θ_M) be a strongly abelian restricted Lie-Rinehart algebra. An *abelian extension* of (A, L, θ_L) by (A, M, θ_M) is a short exact sequence of restricted Lie-Rinehart algebras

$$0 \longrightarrow (A, M, 0) \xrightarrow{\iota} (A, E, \theta_E) \xrightarrow{\pi} (A, L, \theta_L) \longrightarrow 0. \tag{13}$$

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$$\begin{aligned}
[x+u, y+v]_E &:= [x, y]_L + \varphi(x, y) + x \cdot v + y \cdot u, && \forall x, y \in L,\ \forall u, v \in M; \\
(x+u)^{[2]_E} &:= x^{[2]_L} + \omega(x) + x \cdot u, && \forall x \in L,\ \forall u \in M; \\
\theta_E(x+u) &:= \theta_L(x), && \forall x \in L,\ \forall u \in M.
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\end{aligned}$$

Theorem (Bouarroudj-E.-Liu)

Let (A, L, θ_L) be a restricted Lie-Rinehart algebra such that L is projective as A-module. Then, the equivalence classes of abelian extensions of (A, L, θ_L) by a strongly abelian restricted Lie-Rinehart algebra (A, M, θ_M) are classified by $H^2_{\mathrm{LR}}(L; M)$.

Restricted cohomology for Poisson algebras, $p = 2$

Let A be a restricted Poisson algebra in characteristic 2.

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For $n = 0, 1$, we set $C^0_{\mathrm{PA}}(A) := A$ and $C^1_{\mathrm{PA}}(A) := \mathfrak{X}^1(A)$.

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For $n \geq 2$, $C^n_{\mathrm{PA}}(A)$ consists of pairs $(\varphi, \omega) \in C^n_{\mathrm{res}}(L; M)$ where $\varphi \in \mathfrak{X}^n(A)$ and

$$\begin{aligned}\omega(xy, z_2, \cdots, z_{n-1}) = {} & x^2\omega(y, z_2, \cdots, z_{n-1}) + y^2\omega(x, z_2, \cdots, z_{n-1}) \\ & + xy\varphi(x, y, z_2, \cdots, z_{n-1}),\end{aligned} \tag{14}$$

$$\begin{aligned}\omega(x, z_2, \cdots, z_i z_i', \cdots, z_{n-1}) = {} & z_i\omega(x, z_2, \cdots, z_i', \cdots, z_{n-1}) \\ & + z_i'\omega(x, z_2, \cdots, z_i, \cdots, z_{n-1}).\end{aligned} \tag{15}$$

for all $x, y, z_2, \cdots, z_{n-1} \in A$.

Proposition (Bouarroudj-E.-Liu)

Define $\boxed{d^n_{\mathrm{PA}} = d^n_{\mathrm{res}|C^n_{\mathrm{PA}}(A)}.}$ *Then,* $d^n_{\mathrm{PA}}\left(C^n_{\mathrm{PA}}(A)\right) \in C^{n+1}_{\mathrm{PA}}(A)$ *and the spaces* $H^n_{\mathrm{PA}}(A)$ *are well-defined.*

Application: formal deformations

Let $\left(A, \cdot, \{-,-\}, (-)^{\{2\}}\right)$ be a restricted Poisson algebra. Let $A_k^t := A[[t]]/t^{k+1}$ and consider

$$\mu_{(k)} := \{-,-\} + \sum_{i \geq 1}^{k} t^i \mu_i, \quad \text{and} \quad \omega_{(k)} := (-)^{\{2\}} + \sum_{i \geq 1}^{k} t^i \omega_i, \tag{16}$$

where $(\mu_i, \omega_i) \in C^2_{\mathrm{PA}}(A)\ \forall i \geq 1$.

A tuple $\left(A_k^t, \cdot, \mu_{(k)}, \omega_{(k)}\right)$ is called a *formal deformation of order k* of the restricted Poisson algebra $\left(A, \cdot, \{-,-\}, (-)^{\{2\}}\right)$ if $\left(A_k^t, \mu_{(k)}, \omega_{(k)}\right)$ is a restricted Lie algebra.

Since $(\mu_i, \omega_i) \in C^2_{\mathrm{PA}}(A)\ \forall i \geq 1$, $\left(A_k^t, \cdot, \mu_{(k)}, \omega_{(k)}\right)$ is a restricted Poisson algebra.

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Since $(\mu_i, \omega_i) \in C^2_{\mathrm{PA}}(A)$ $\forall i \geq 1$, $\left(A_k^t, \cdot, \mu_{(k)}, \omega_{(k)}\right)$ is a restricted Poisson algebra.

Theorem (Bouarroudj-E.-Liu)

Let $\left(A, \cdot, \{-,-\}, (-)^{\{2\}}\right)$ be a restricted Poisson algebra. Then, the second cohomology space $\mathrm{H}^2_{\mathrm{PA}}(A)$ classifies (up to equivalence) the deformations of $\left(A, \cdot, \{-,-\}, (-)^{\{2\}}\right)$ of order 1.

Example: Heisenberg

Consider the 3-dimensional restricted Heisenberg Lie algebra $\mathfrak{h}$ spanned by elements e_1, e_2, e_3 with

$$[e_1, e_2] = e_3, \quad (-)^{[2]} = 0, \quad e_1 e_2 = e_3.$$

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$$\mathrm{H}^1_{\mathrm{PA}}(\mathfrak{h}) = \mathsf{Span}\{e_1 \otimes e_1^* + e_2 \otimes e_2^*,\ e_1 \otimes e_1^* + e_3 \otimes e_3^*\}.$$

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$$\mathrm{H}^2_{\mathrm{PA}}(\mathfrak{h}) = \big\{(\varphi_1, 0), (\varphi_2, 0)\big\},$$

where

$$\varphi_1 = e_1 \otimes (e_1^* \wedge e_2^*) + e_3 \otimes (e_2^* \wedge e_3^*);$$
$$\varphi_2 = e_2 \otimes (e_1^* \wedge e_2^*) + e_3 \otimes (e_1^* \wedge e_3^*).$$

Example: analogs of "classical Poisson" $\mathfrak{po}_\Pi(2n, \underline{N})$

Let $k \geq 1$. The associative commutative algebra of divided powers in k variables $x := (x_1, \cdots, x_k)$ is defined for $\underline{N} := (n_1, \cdots, n_k)$, where $n_s \geq 0$, $\forall\ 1 \leq s \leq k$, by

$$\mathbb{K}(x; \underline{N}) := \mathrm{Span}\left\{x^{(\underline{i})} := x_1^{(i_1)} \cdots x_k^{(i_k)},\ (\underline{i}) = (i_1, \cdots, i_k),\ 0 \leq i_s \leq p^{n_s} - 1\right\}.$$

The multiplication is given by

$$x^{(\underline{i})} x^{(\underline{j})} = \binom{\underline{i} + \underline{j}}{\underline{i}} x^{(\underline{i}+\underline{j})}, \text{ where } \binom{\underline{i} + \underline{j}}{\underline{i}} := \prod_{s=1}^{k} \binom{i_s + j_s}{i_s}. \tag{17}$$

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For $n \geq 1$, define

$$\mathfrak{po}_\Pi(2n, \underline{N}) := \{f \in \mathbb{K}[p_i, q_i; \underline{N}],\ 1 \leq i \leq n\},$$

endowed with the Lie bracket

$$\{f, g\}_\Pi := \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right). \tag{18}$$

Example: analogs of "classical Poisson" $\mathfrak{po}_\Pi(2n, \underline{N})$

Lemma

The space $\mathfrak{po}_\Pi(2n, \underline{1})$ *is a restricted Poisson algebra with the bracket* $\{-,-\}_\Pi$ *and the 2-map*

$$(p_i q_j)^{\{2\}} := \delta_{i,j} p_i q_j.$$

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Proposition

We have $H^1_{\mathrm{PA}}(\mathfrak{po}_{\Pi}(2, \underline{1})) = 0$ *and* $H^2_{\mathrm{PA}}(\mathfrak{po}_{\Pi}(2, \underline{1})) = \mathit{Span}\{(0, \omega)\}$, *where*

$$\omega(1) = 1, \ \omega(p_1) = \omega(q_1) = \omega(p_1 q_1) = 0.$$

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To compare:

$$\dim H^1_{\mathrm{CE}}(\mathfrak{po}_\Pi(2, \underline{1})) = 4;$$
$$\dim H^2_{\mathrm{CE}}(\mathfrak{po}_\Pi(2, \underline{1})) = 6.$$

A comparison between the cohomologies

Kähler differentials.

Let A be an *unital* associative commutative $\mathbb{K}$-algebra. The module of Kähler differentials of A, denoted by $\Omega^1(A)$, is the free A-module with

- **generators:** $\mathrm{d}\,x,\ x \in A$;
- **relations:**

$$\mathrm{d}(xy) = y\,\mathrm{d}\,x + x\,\mathrm{d}\,y;\ \mathrm{d}(x+y) = \mathrm{d}\,x + \mathrm{d}\,y;\ \mathrm{d}\,\lambda = 0, \quad \forall x, y \in A,\ \forall \lambda \in \mathbb{K}\,.$$

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- **universal property:**

$$\begin{array}{ccc} A & \xrightarrow{\ \mathrm{d}\ } & \Omega^1(A)\ , \\ {\scriptstyle \text{derivation } D}\downarrow & \swarrow {\scriptstyle \exists!\ \hat{D}\ A\text{-linear}} & \\ A & & \end{array} \tag{19}$$

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where A is seen as a module over itself using its multiplication.

By this universal property, we have a natural isomorphism of A-modules

$$\mathrm{Der}(A) \cong \mathrm{Hom}_A(\Omega^1(A), A). \tag{20}$$

Huebschmann's theorem, $p = 2$

Let $(A, \cdot, \{-,-\})$ be an ordinary Poisson algebra and $\Omega^1(A)$ its module of Kähler differentials, seen as a Lie algebra with the bracket

$$[x \,\mathrm{d}\, u, y \,\mathrm{d}\, v]_{\Omega^1(A)} := x\{u, y\} \,\mathrm{d}\, v + y\{x, v\} \,\mathrm{d}\, u + xy \,\mathrm{d}\{u, v\}, \quad \forall x, y, u, v \in A. \tag{21}$$

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Then, $\left(A, \Omega^1(A), \pi^\sharp\right)$ is a Lie-Rinehart algebra, with

$$\pi^\sharp : \Omega^1(A) \to \mathrm{Hom}_A\left(\Omega^1(A), A\right) \cong \mathrm{Der}(A), \quad x\,\mathrm{d}\,u \mapsto x\{u,-\}. \tag{22}$$

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Theorem (Bouarroudj-E.-Liu)

Let $(A, \cdot, \{-,-\}, (-)^{\{2\}})$ be a restricted Poisson algebra in characteristic 2, and suppose that the module of Kähler differentials is free as an A-module. Define a map $(-)^{[2]} : \Omega^1(A) \to \Omega^1(A)$ by

$$(x \,\mathrm{d}\, u)^{[2]} = x^2 \,\mathrm{d}(u^{\{2\}}) + x\{u,x\} \,\mathrm{d}\, u, \quad \forall\, x \,\mathrm{d}\, u \in \Omega^1(A). \tag{23}$$

Then, $\left(\Omega^1(A), [-,-]_{\Omega^1(A)}, (-)^{[2]}\right)$ is a restricted Lie algebra and $\left(A, \Omega^1(A), \pi^\sharp\right)$ is a restricted Lie-Rinehart algebra.

Kähler forms

Let $k \geq 0$ and consider the A-modules $\Omega^k(A) := \wedge_A^k \Omega^1(A)$, $k \geq 1$ and $\Omega^0(A) := A$, and

$$\Omega^\bullet(A) := \bigoplus_{k \geq 0} \Omega^k(A).$$

The map $\mathrm{d} : A \to \Omega^1(A)$ extends a $\mathbb{K}$-linear map $\wedge^\bullet \mathrm{d} : \wedge^\bullet A \to \Omega^\bullet(A)$ by the formula

$$\wedge^\bullet \mathrm{d}(u_1 \wedge \cdots \wedge u_k) := \mathrm{d}\, u_1 \wedge \cdots \wedge \mathrm{d}\, u_k, \quad \forall u_1, \cdots u_k \in A. \tag{24}$$

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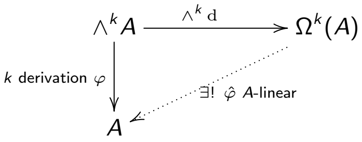
Universal property:



(25)

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Universal property:

$$\begin{array}{ccc} \wedge^k A & \xrightarrow{\ \wedge^k \mathrm{d}\ } & \Omega^k(A) \\ {\scriptstyle k \text{ derivation } \varphi}\big\downarrow & \swarrow {\scriptstyle \exists!\ \hat{\varphi}\ A\text{-linear}} & \\ A & & \end{array} \tag{25}$$

As a consequence, we have a natural isomorphism of A-modules:

$$\mathfrak{X}^k(A) \cong \mathrm{Hom}_A(\Omega^k(A), A) \tag{26}$$

and the map $\varphi \to \hat{\varphi}$ induces a natural isomorphism

$$\mathfrak{X}^\bullet(A) \cong \bigoplus_{k \in \mathbb{N}} \mathrm{Hom}_A(\Omega^k(A), A). \tag{27}$$

The comparison theorem, 1/2

Proposition (Bouarroudj-E.-Liu)

For $k \geq 2$, let $(\varphi, \omega) \in C^k_{\mathrm{PA}}(A)$. Then, there exists an unique map

$$\hat{\omega} : \Omega^1(A) \otimes \Omega^{k-2}(A) \to A$$

such that

$$(\hat{\varphi}, \hat{\omega}) \in C^k_{\mathrm{LR}}(\Omega^k(A); A) \text{ and } \hat{\omega} \circ (\mathrm{d} \otimes \wedge^{k-2}\, \mathrm{d}) = \omega,$$

that is, the following diagram commutes:

$$\begin{array}{ccc} A \otimes \wedge^{k-2} A & \xrightarrow{\mathrm{d} \otimes \wedge^{k-2} \mathrm{d}} & \Omega^1(A) \otimes \Omega^{k-2}(A) \\ \downarrow{\scriptstyle \omega} & \swarrow{\scriptstyle \exists!\ \hat{\omega}} & \\ A & & \end{array} \tag{28}$$

where A is seen as a module over itself using its multiplication.

The comparison theorem, 2/2

Theorem (Bouarroudj-E.-Liu)

Let $(A, \cdot, \{-,-\}, (-)^{\{2\}})$ be a restricted Poisson algebra in characteristic 2. If the module of the Kähler differential $\Omega^1(A)$ is free, then the cohomology complex

$$\left(\bigoplus_{n\geq 0} C^n_{\mathrm{PA}}(A), \mathrm{d}^n_{\mathrm{PA}}\right)$$

for the restricted Poisson algebra A is isomorphic to the cohomology complex

$$\left(\bigoplus_{n\geq 0} C^n_{\mathrm{LR}}\left(\Omega^1(A); A\right), \mathrm{d}^n_{\mathrm{LR}}\right)$$

for the restricted Lie-Rinehart algebra $\left(A, \Omega^1(A), \pi^\sharp\right)$.

Thank you for your attention

Main reference:

S. Bouarroudj, Q. Ehret, J. Liu,
Cohomology of restricted Poisson algebras in characteristic 2,
arXiv:2504.07601.