Lie-Rinehart (Super)algebras

Differential Geometry, Contact Geometry, Dynamical Systems and beyond
Conference in memory of Robert Lutz (1943-2020)

Quentin Ehret

Supervisor: Abdenacer Makhlouf

28 June 2022

IRIMAS UNIVERSITÉ HAUTE-ALSACE
Institut de Recherche en Informatique, Mathématiques, Automatique et Signal
Département de Mathématiques

Lie algebroids

M a smooth manifold; $T(M)$ tangent bundle of M;
$X(M)$ Lie algebra of smooth vector fields on M.

A **Lie algebroid** is a smooth vector bundle $(\pi, \mathfrak{g})$ over M, with $\pi : \mathfrak{g} \longrightarrow M$ endowed with

- a smooth vector bundle morphism $\alpha : \mathfrak{g} \longrightarrow T(M)$
- a Lie structure on $\Gamma\mathfrak{g}$, the vector space of smooth sections of $\mathfrak{g}$

Lie algebroids

M a smooth manifold; $T(M)$ tangent bundle of M;
$X(M)$ Lie algebra of smooth vector fields on M.

A **Lie algebroid** is a smooth vector bundle $(\pi, \mathfrak{g})$ over M, with $\pi : \mathfrak{g} \longrightarrow M$ endowed with

- a smooth vector bundle morphism $\alpha : \mathfrak{g} \longrightarrow T(M)$
- a Lie structure on $\Gamma\mathfrak{g}$, the vector space of smooth sections of $\mathfrak{g}$

such that:

1. $\Gamma(\alpha) : \Gamma\mathfrak{g} \longrightarrow X(M)$ is a Lie morphism;

Lie algebroids

M a smooth manifold; $T(M)$ tangent bundle of M;
$X(M)$ Lie algebra of smooth vector fields on M.

A **Lie algebroid** is a smooth vector bundle $(\pi, \mathfrak{g})$ over M, with $\pi : \mathfrak{g} \longrightarrow M$ endowed with

- a smooth vector bundle morphism $\alpha : \mathfrak{g} \longrightarrow T(M)$
- a Lie structure on $\Gamma\mathfrak{g}$, the vector space of smooth sections of $\mathfrak{g}$

such that:

1. $\Gamma(\alpha) : \Gamma\mathfrak{g} \longrightarrow X(M)$ is a Lie morphism;
2. $[x, fy] = f[x, y] + \Gamma(\alpha)(x)(f)y, \ f \in C^\infty(M), \ x, y \in \Gamma\mathfrak{g}$.

Lie algebroids

Teaser:

Lie algebroids over M are precisely Lie-Rinehart algebras over $C^\infty(M)$ such that the Lie algebra L is finitely generated and projective as $C^\infty(M)$-module.

Definitions

$\mathbb{K}$ is a field of characteristic 0.

Definition

A **Lie superalgebra** *L is a $\mathbb{Z}_2$-graded $\mathbb{K}$-module $L = L_0 \bigoplus L_1$ endowed with a bracket $[\cdot,\cdot]$ (called super Lie bracket) satisfying, for homogeneous elements $x, y, z \in L$:*

- $|[x,y]| = |x| + |y|$;
- $[x,y] = -(-1)^{|x||y|}[y,x]$ *(super-skewsymmetry);*
- $(-1)^{|x||z|}[x,[y,z]]+(-1)^{|z||y|}[z,[x,y]]+(-1)^{|x||y|}[y,[z,x]] = 0$ *(super-Jacobi).*

Definitions

Definition

An **associative superalgebra** *is a* $\mathbb{Z}_2$*-graded* $\mathbb{K}$*-module* $A = A_0 \oplus A_1$ *endowed with a bilinear map* $A \times A \longrightarrow A$ *such that* $(ab)c = a(bc)$ *for all* $a, b, c \in A$*, and* $A_i A_j \subset A_{i+j}$.

It is called **supercommutative** *if* $ab = (-1)^{|a||b|} ba, \ \forall a, b$ *homogeneous in* A.

Definitions

Definition

Let A be an algebra. A map $D : A \longrightarrow A$ is a **superderivation** *(of degree $|D|$) of A if D is a $\mathbb{Z}_2$-graded linear map and if the super-Leibniz condition is satisfied:*

$$D(ab) = D(a)b + (-1)^{|a||D|} aD(b) \quad \forall a, b \in A.$$

We denote by $\mathrm{Der}(A)$ *the vector superspace of superderivations of A.*

Definitions

Definition

A **Lie-Rinehart superalgebra** *is a pair* (A, L)*, where*

- L *is a Lie superalgebra over* $\mathbb{K}$*, endowed with a bracket* $[\cdot,\cdot]$*;*
- A *is an associative and supercommutative* $\mathbb{K}$*-superalgebra,*

such that, for $x, y \in L$ *and* $a, b \in A$*:*

- *There is an action* $A \times L \longrightarrow L,\ (a, x) \longmapsto a \cdot x$*, making* L *an* A*-module;*
- *There is a map* $\rho : L \longrightarrow Der(A),\ x \longmapsto \rho_x$*, which is both a Lie algebra morphism and a* A*-module morphism, called* **anchor map***;*
- $[x, a \cdot y] = \rho_x(a) \cdot y + (-1)^{|a||x|} a \cdot [x, y]$.

Examples

- **Trivial Lie-Rinehart superalgebra.** Let A be an associative unital superalgebra and L be a Lie superalgebra. Then the pair (A, L) can always be endowed with a Lie-Rinehart superalgebra structure with the trivial action and the zero anchor.

Examples

- **Trivial Lie-Rinehart superalgebra.** Let A be an associative unital superalgebra and L be a Lie superalgebra. Then the pair (A, L) can always be endowed with a Lie-Rinehart superalgebra structure with the trivial action and the zero anchor.
- **Lie superalgebra of superderivations.** Let A be an associative unital superalgebra, and $L = \mathrm{Der}(A)$ be its superderivations superalgebra. Then $(A, \mathrm{Der}(A))$ is a Lie-Rinehart superalgebra, with $\rho = \mathrm{id}$.

Classification: an example

(1|1, 1|1)-type: $(\alpha_i \in \mathbb{C},\ \alpha_i \neq 0)$

A	L	Action	Anchor
$A^1_{1\|1}$	$L^1_{1\|1}$	$e^1_1 \cdot f^1_1 = \alpha_1 f^0_1$	null
	$L^2_{1\|1}$	trivial	$\rho(f^0_1)(e^1_1) = \alpha_2 e^1_1$
		$e^1_1 \cdot f^1_1 = \alpha_3 f^0_1$	$\rho(f^0_1)(e^1_1) = -e^1_1,\ \rho(f^1_1)(e^1_1) = -\alpha_3 e^0_1$
		$e^1_1 \cdot f^0_1 = \alpha_4 f^1_1$	$\rho(f^0_1)(e^1_1) = e^1_1$
	$L^3_{1\|1}$	trivial	$\rho(f^0_1)(e^1_1) = \alpha_5 e^1_1$
		$e^1_1 \cdot f^0_1 = \alpha_6 f^1_1$	null

Classification: an example

(1|1, 1|1)-type: $(\alpha_i \in \mathbb{C},\ \alpha_i \neq 0)$

A	L	Action	Anchor
$A^1_{1\|1}$	$L^1_{1\|1}$	$e^1_1 \cdot f^1_1 = \alpha_1 f^0_1$	null
	$L^2_{1\|1}$	trivial	$\rho(f^0_1)(e^1_1) = \alpha_2 e^1_1$
		$e^1_1 \cdot f^1_1 = \alpha_3 f^0_1$	$\rho(f^0_1)(e^1_1) = -e^1_1,\ \rho(f^1_1)(e^1_1) = -\alpha_3 e^0_1$
		$e^1_1 \cdot f^0_1 = \alpha_4 f^1_1$	$\rho(f^0_1)(e^1_1) = e^1_1$
	$L^3_{1\|1}$	trivial	$\rho(f^0_1)(e^1_1) = \alpha_5 e^1_1$
		$e^1_1 \cdot f^0_1 = \alpha_6 f^1_1$	null

$\rightsquigarrow$ We have obtained all Lie-Rinehart superalgebras structures on pairs (A, L) with $\dim(A) \leq 2$ and $\dim(L) \leq 4$.

Deformation theory: Super-multiderivations

Let $(A, L[\cdot,\cdot], \rho)$ be a Lie-Rinehart superalgebra and M an A-module.

Definition (Super-multiderivations space)

We define $\mathrm{Der}^n(M, M)$ *as the space of multilinear maps*

$$f : M^{\otimes n+1} \longrightarrow M$$

such that it exists $\sigma_f : M^{\otimes n} \longrightarrow \mathrm{Der}(A)$ *(called symbol map),*

$$\begin{aligned} f(x_1, \ldots, x_{n+1}) &= -(-1)^{|x_i||x_{i+1}|} f(x_1, \ldots, x_{i+1}, x_i, \ldots, x_{n+1}), \\ f(x_1, \ldots x_i, \ldots a \cdot x_{n+1}) &= (-1)^{|a|(|f|+|x_1|+\ldots+|x_n|)} a \cdot f(x_1, \ldots, x_{n+1}) \\ &\quad + \sigma_f(x_1, \ldots, x_n)(a)(x_{n+1}), \ \forall a \in A. \end{aligned}$$

Deformation theory: Super-multiderivations

$$\mathrm{Der}^*(M, M) = \bigoplus_{n \geq -1} \mathrm{Der}^n(M, M), \text{ with } \mathrm{Der}^{-1}(M, M) = M.$$

Deformation theory: Super-multiderivations

$$\mathrm{Der}^*(M,M) = \bigoplus_{n\geq -1} \mathrm{Der}^n(M,M), \text{ with } \mathrm{Der}^{-1}(M,M) = M.$$

Lie structure: $f \in \mathrm{Der}^p(M,M)$ and $g \in \mathrm{Der}^q(M,M)$:

$$[f,g] = f \circ g - (-1)^{pq} g \circ f,$$

with symbol map $\sigma_{[f,g]} = \sigma_f \circ g - (-1)^{pq}\sigma_g \circ f + [\sigma_f, \sigma_g]$,

Deformation theory: Super-multiderivations

$$\mathrm{Der}^*(M, M) = \bigoplus_{n \geq -1} \mathrm{Der}^n(M, M), \text{ with } \mathrm{Der}^{-1}(M, M) = M.$$

Lie structure: $f \in \mathrm{Der}^p(M, M)$ and $g \in \mathrm{Der}^q(M, M)$:

$$[f, g] = f \circ g - (-1)^{pq} g \circ f,$$
$$\text{with symbol map } \sigma_{[f,g]} = \sigma_f \circ g - (-1)^{pq} \sigma_g \circ f + [\sigma_f, \sigma_g],$$

with

$$(f \circ g)(x_1, \dots, x_{p+q+1}) = \sum_{\tau \in Sh(q+1,p)} \varepsilon(\tau, x_1, \dots, x_{p+q+1})$$
$$\times f\left(g(x_{\tau(1)}, \dots, x_{\tau(q+1)}), x_{\tau(q+2)}, \dots, x_{\tau(p+q+1)}\right).$$

Deformation theory: Super-multiderivations

Proposition

There is a one-to-one correspondence between Lie-Rinehart superstructures on (A, L) *and elements* $m \in \mathrm{Der}^1(L, L)$ *such that* $[m, m] = 0$.

Deformation theory: Super-multiderivations

Proposition

There is a one-to-one correspondence between Lie-Rinehart superstructures on (A, L) and elements $m \in \mathrm{Der}^1(L, L)$ such that $[m, m] = 0$.

$\rightsquigarrow$ with that in mind, we can construct a deformation theory by identifying Lie-Rinehart superstructures on (A, L) and the corresponding element $m \in \mathrm{Der}^1(L, L)$.

Deformation theory: Deformation cohomology

Cochains space.

$$C^n_{def}(L, L) := \mathrm{Der}^{n-1}(L, L)$$
$$C^*_{def}(L, L) := \bigoplus_{n \geq 0} C^n_{def}(L, L).$$

Deformation theory: Deformation cohomology

Cochains space.

$$C^n_{def}(L, L) := \mathrm{Der}^{n-1}(L, L)$$
$$C^*_{def}(L, L) := \bigoplus_{n\geq 0} C^n_{def}(L, L).$$

We endow it with a differential operator

$$\delta : C^n_{def}(L, L) \longrightarrow C^{n+1}_{def}(L, L),$$
$$D \longmapsto [m, D].$$

Deformation theory: Deformation cohomology

Cochains space.

$$C^n_{def}(L, L) := \mathrm{Der}^{n-1}(L, L)$$
$$C^*_{def}(L, L) := \bigoplus_{n \geq 0} C^n_{def}(L, L).$$

We endow it with a differential operator

$$\delta : C^n_{def}(L, L) \longrightarrow C^{n+1}_{def}(L, L),$$
$$D \longmapsto [m, D].$$

Proposition

$(C^*_{def}(L, L), \delta)$ *is a cochain complex.*

Deformation theory: main results

Definition

Let $(A, L, [\cdot,\cdot], \rho)$ *be a Lie-Rinehart superalgebra, and let* $m \in \mathrm{Der}^1(L, L)$ *be the corresponding super-multiderivation.*

$$\begin{aligned} m_t : L \times L &\longrightarrow L[[t]] \\ (x, y) &\longmapsto \sum_{i \geq 0} t^i m_i(x, y), \quad m_0 = m,\ m_i \in \mathrm{Der}^1(L, L), \end{aligned}$$

Moreover, m_t *must verify* $[m_t, m_t] = 0$*, the bracket being the* $\mathbb{Z}$*-graded bracket on* $C^*_{def}(L[[t]], L[[t]])$*.*

Deformation theory: main results

Theorem

Let m_t be a deformation of a Lie-Rinehart superalgebra (A, L). Then the infinitesimal m_1 is a 2-cocycle with respect to the deformation cohomology.

Deformation theory: main results

Theorem

Let m_t be a deformation of a Lie-Rinehart superalgebra (A, L). Then the infinitesimal m_1 is a 2-cocycle with respect to the deformation cohomology.

Theorem

Any non-trivial deformation of $m \in Der^1(L, L)$ is equivalent to a deformation whose infinitesimal is not a coboundary.

Deformation theory: main results

Theorem

Let m_t be a deformation of a Lie-Rinehart superalgebra (A, L). Then the infinitesimal m_1 is a 2-cocycle with respect to the deformation cohomology.

Theorem

Any non-trivial deformation of $m \in Der^1(L, L)$ is equivalent to a deformation whose infinitesimal is not a coboundary.

$\Longrightarrow$ If $H^2_{def} = 0$, any deformation is equivalent to the trivial deformation.

A quick look into the world of positive characteristic.

Let $\mathbb{F}$ be a field of prime characteristic p.

Definition (Jacobson)

A **restricted Lie algebra** *is a Lie algebra L endowed with an application $(\cdot)^{[p]} : L \longrightarrow L$ such that*

1. $(\lambda x)^{[p]} = \lambda^p x^{[p]}$, $x \in L$, $\lambda \in \mathbb{F}$;

A quick look into the world of positive characteristic.

Let $\mathbb{F}$ be a field of prime characteristic p.

Definition (Jacobson)

A **restricted Lie algebra** *is a Lie algebra L endowed with an application $(\cdot)^{[p]} : L \longrightarrow L$ such that*

1. $(\lambda x)^{[p]} = \lambda^p x^{[p]}$, $x \in L$, $\lambda \in \mathbb{F}$;
2. $\left[x, y^{[p]}\right] = [[\dots[x, \overbrace{y], y], \dots, y}^{p \text{ termes}}]$;

A quick look into the world of positive characteristic.

Let $\mathbb{F}$ be a field of prime characteristic p.

Definition (Jacobson)

A **restricted Lie algebra** *is a Lie algebra L endowed with an application $(\cdot)^{[p]} : L \longrightarrow L$ such that*

1. $(\lambda x)^{[p]} = \lambda^p x^{[p]}$, $x \in L$, $\lambda \in \mathbb{F}$;
2. $\left[x, y^{[p]}\right] = [[\dots[x, \overbrace{y], y], \dots, y}^{p \text{ termes}}]$;
3. $(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x,y)$,

with $is_i(x,y)$ the coefficient of Z^{i-1} in $\mathrm{ad}^{p-1}_{Zx+y}(x)$. Such an application $(-)^{[p]} : L \longrightarrow L$ is called a p-map.

Restricted cohomology of restricted Lie algebras

Definition (Restricted 2-cochains)

Let $\varphi \in C^2_{CE}(L, M)$ *and* $\omega : L \longrightarrow M$. *Then* ω **has the $(*)$-property w.r.t** φ *if*

1. $\omega(\lambda x) = \lambda^p \omega(x),\ \lambda \in \mathbb{F},\ x \in L;$

Restricted cohomology of restricted Lie algebras

Definition (Restricted 2-cochains)

Let $\varphi \in C^2_{CE}(L, M)$ *and* $\omega : L \longrightarrow M$*. Then* ω **has the (*)-property w.r.t** φ *if*

1. $\omega(\lambda x) = \lambda^p \omega(x), \ \lambda \in \mathbb{F}, \ x \in L;$
2. $\omega(x+y) = \omega(x) + \omega(y) +$

$$\sum_{\substack{x_i = x \text{ or } y \\ x_1 = x, \ x_2 = y}} \frac{1}{\pi(x)} \sum_{k=0}^{p-2} (-1)^k x_p \dots x_{p-k+1} \varphi([[\dots[x_1, x_2], x_3] \dots, x_{p-k-1}], x_{p-k}),$$

with $x, y \in L$, $\pi(x)$ *the number of factors* x_i *equal to* x.

Restricted cohomology of restricted Lie algebras

Definition (Restricted 2-cochains)

Let $\varphi \in C^2_{CE}(L,M)$ *and* $\omega : L \longrightarrow M$. *Then* ω **has the (∗)-property w.r.t** φ *if*

1. $\omega(\lambda x) = \lambda^p \omega(x),\ \lambda \in \mathbb{F},\ x \in L;$
2. $\omega(x+y) = \omega(x) + \omega(y) +$
$$\sum_{\substack{x_i = x \text{ or } y \\ x_1 = x,\ x_2 = y}} \frac{1}{\pi(x)} \sum_{k=0}^{p-2} (-1)^k x_p \ldots x_{p-k+1} \varphi([[\ldots[x_1,x_2],x_3]\ldots,x_{p-k-1}],x_{p-k}),$$

with $x,y \in L$, $\pi(x)$ *the number of factors* x_i *equal to* x.*We then define*

$$C^2_*(L,M) = \Big\{(\varphi,\omega),\ \varphi \in C^2_{CE}(L,M),\ \omega \text{ has the } (*)\text{-property w.r.t } \varphi\Big\}$$

Restricted cohomology of restricted Lie algebras

- A **restriced 2-cocycle** is an element $(\alpha, \beta) \in C^2_*(L, M)$ such that
  1. α is an ordinary Chevalley-Eilenberg 2-cocycle;
  2. $\alpha\left(x, y^{[p]}\right) - \sum_{i+j=p-1} (-1)^i y^i \alpha\left([x, \underbrace{y, ..., y}_{j \text{ terms}}], y\right) + x\beta(y) = 0.$

Restricted cohomology of restricted Lie algebras

- A **restriced 2-cocycle** is an element $(\alpha, \beta) \in C^2_*(L, M)$ such that
  1. α is an ordinary Chevalley-Eilenberg 2-cocycle;
  2. $\alpha\left(x, y^{[p]}\right) - \sum_{i+j=p-1} (-1)^i y^i \alpha\left([x, \underbrace{y, \ldots, y}_{j \text{ terms}}], y\right) + x\beta(y) = 0.$
- A **restriced 2-coboundary** is an element $(\alpha, \beta) \in C^2_*(L, M)$ such that $\exists \varphi \in \mathrm{Hom}(L, M)$,
  1. $\alpha(x, y) = \varphi([x, y]) - x\varphi(y) + y\varphi(x);$
  2. $\beta(x) = \varphi\left(x^{[p]}\right) - x^{p-1}\varphi(x).$

A quick look into the world of positive characteristic.

Let A be an associative commutative algebra, $D \in \mathrm{Der}(A)$ and $a \in A$. In characteristic p, we have:

$$(aD)^p = a^p D^p + (aD)^{p-1}(a)D.$$

A quick look into the world of positive characteristic.

Definition

Let A be an associative commutative algebra over a field $\mathbb{F}$ of characteristic p. Then (A, L) is a **restricted Lie-Rinehart algebra** *if*

1. (A, L) *is a Lie-Rinehart algebra, with anchor map* $\rho : L \longrightarrow \mathrm{Der}(A)$;
2. $\left(L, (\cdot)^{[p]}\right)$ *is a restricted Lie algebra;*
3. $\rho(x^{[p]}) = \rho(x)^p$;
4. $(ax)^{[p]} = a^p x^{[p]} + \rho(ax)^{p-1}(a)x, \ a \in A, \ x \in L.$

Infinitesimal Deformations:

$$m_t = [\cdot,\cdot] + t m_1,\ m_1 \text{ multiderivation ;} \tag{1}$$

$$\sigma_t = \rho + t\sigma_1,\ \ \sigma_1 \text{ symbol map with respect to } m_1; \tag{2}$$

$$(\cdot)^{[p]_t} = (\cdot)^{[p]} + t\omega_1,\ \omega_1 : L \to L. \tag{3}$$

Infinitesimal Deformations:

$$m_t = [\cdot,\cdot] + t m_1,\ \ m_1 \text{ multiderivation ;} \quad (1)$$

$$\sigma_t = \rho + t\sigma_1,\ \ \sigma_1 \text{ symbol map with respect to } m_1; \quad (2)$$

$$(\cdot)^{[p]_t} = (\cdot)^{[p]} + t\omega_1,\ \omega_1 : L \to L. \quad (3)$$

We obtain the following deformations equations:

$$[x, m_1(y,z)] + [y, m_1(z,x)] + [z, m_1(x,y)] \quad (4)$$
$$+ m_1(x,[y,z]) + m_1(y,[z,x]) + m_1(z,[x,y]) = 0$$

$$\Big([x,\omega_1(y)] + m_1(x, y^{[p]})\Big) = \sum_{i+j=p-1} (-1)^i y^i m_1([x, \overbrace{y,\dots,y}^{j}], y) \quad (5)$$

Proposition

Equations (4) and (5) imply that (m_1,ω_1) is a restricted 2-cocycle.

Infinitesimal Deformations:

$$m_t = [\cdot,\cdot] + tm_1,\ m_1 \text{ multiderivation ;} \quad (1)$$

$$\sigma_t = \rho + t\sigma_1,\ \ \sigma_1 \text{ symbol map with respect to } m_1; \quad (2)$$

$$(\cdot)^{[p]_t} = (\cdot)^{[p]} + t\omega_1,\ \omega_1 : L \to L. \quad (3)$$

We also obtain the new following equations:

$$\rho(\omega_1(x)) + \sigma_1\left(x^{[p]}\right) = \sum_{i=0}^{p-1} \rho(x)^i \circ \sigma_1(x) \circ \rho(x)^{p-1-i} \quad (6)$$

$$\omega_1(ax) - a^p\omega_1(x) = \sum_{i=0}^{p-2} \rho(ax)^i \circ \sigma_1(ax) \circ \rho(ax)^{p-2-i}(a)x \quad (7)$$

Ongoing works: Finding cohomological interpretations of Equations (6) and (7).

Thank you for your attention.