Deformations of Lie-Rinehart (Super)algebras

Quentin Ehret

New Trends in Hopf Algebras and Monoidal Categories, Torino

September 2022

DIPARTIMENTO
DI MATEMATICA
GIUSEPPE PEANO
UNIVERSITÀ DI TORINO

IRIMAS UNIVERSITÉ HAUTE-ALSACE
Institut de Recherche en Informatique, Mathématiques, Automatique et Signal
Département de Mathématiques

1. Introduction
2. Deformations of Lie-Rinehart superalgebras
3. The restricted side of the story: positive characteristic.

Definitions

$\mathbb{K}$ is a field of characteristic 0.

Definition

A **Lie superalgebra** *L is a $\mathbb{Z}_2$-graded $\mathbb{K}$-module $L = L_0 \bigoplus L_1$ endowed with a bracket $[\cdot, \cdot]$ (called super Lie bracket) satisfying, for homogeneous elements $x, y, z \in L$:*

- $|[x, y]| = |x| + |y|$;
- $[x, y] = -(-1)^{|x||y|}[y, x]$ *(super-skewsymmetry);*
- $(-1)^{|x||z|}[x, [y, z]] + (-1)^{|z||y|}[z, [x, y]] + (-1)^{|x||y|}[y, [z, x]] = 0$ *(super-Jacobi).*

Definitions

Definition

An **associative superalgebra** *is a* $\mathbb{Z}_2$*-graded* $\mathbb{K}$*-module* $A = A_0 \oplus A_1$ *endowed with a bilinear map* $A \times A \longrightarrow A$ *such that* $(ab)c = a(bc)$ *for all* $a, b, c \in A$, *and* $A_i A_j \subset A_{i+j}$.

It is called **supercommutative** *if* $ab = (-1)^{|a||b|} ba, \ \forall a, b$ *homogeneous in* A.

Definitions

Definition

Let A be a superalgebra. A map $D : A \longrightarrow A$ is a **superderivation** *(of degree $|D|$) of A if D is a $\mathbb{Z}_2$-graded linear map and if the super-Leibniz condition is satisfied:*

$$D(ab) = D(a)b + (-1)^{|a||D|} aD(b) \quad \forall a, b \in A.$$

We denote by $\mathrm{Der}(A)$ the vector superspace of superderivations of A.

Definitions

Definition

A **Lie-Rinehart superalgebra** *is a pair* (A, L)*, where*

- L *is a Lie superalgebra over* $\mathbb{K}$*, endowed with a bracket* $[\cdot,\cdot]$*;*
- A *is an associative and supercommutative* $\mathbb{K}$*-superalgebra,*

such that, for $x, y \in L$ *and* $a, b \in A$*:*

- *There is an action* $A \times L \longrightarrow L, \ (a, x) \longmapsto a \cdot x$*, making* L *an* A*-module;*
- *There is a map* $\rho : L \longrightarrow Der(A), \ x \longmapsto \rho_x$*, which is both a Lie algebra morphism and a* A*-module morphism, called* **anchor map***;*
- $[x, a \cdot y] = \rho_x(a) \cdot y + (-1)^{|a||x|} a \cdot [x, y]$.

Examples

- **Trivial Lie-Rinehart superalgebra.** Let A be an associative unital superalgebra and L be a Lie superalgebra. Then the pair (A, L) can always be endowed with a Lie-Rinehart superalgebra structure with the trivial action and the zero anchor.

Examples

- **Trivial Lie-Rinehart superalgebra.** Let A be an associative unital superalgebra and L be a Lie superalgebra. Then the pair (A, L) can always be endowed with a Lie-Rinehart superalgebra structure with the trivial action and the zero anchor.

- **Lie superalgebra of superderivations.** Let A be an associative unital superalgebra, and $L = \mathrm{Der}(A)$ be its superderivations superalgebra. Then $(A, \mathrm{Der}(A))$ is a Lie-Rinehart superalgebra, with $\rho = \mathrm{id}$.

Classification: an example

(1|1, 1|1)-type: $(\alpha_i \in \mathbb{C},\ \alpha_i \neq 0)$

A	L	Action	Anchor
$A^1_{1\|1}$	$L^1_{1\|1}$	$e^1_1 \cdot f^1_1 = \alpha_1 f^0_1$	null
	$L^2_{1\|1}$	trivial	$\rho(f^0_1)(e^1_1) = \alpha_2 e^1_1$
		$e^1_1 \cdot f^1_1 = \alpha_3 f^0_1$	$\rho(f^0_1)(e^1_1) = -e^1_1,\ \rho(f^1_1)(e^1_1) = -\alpha_3 e^0_1$
		$e^1_1 \cdot f^0_1 = \alpha_4 f^1_1$	$\rho(f^0_1)(e^1_1) = e^1_1$
	$L^3_{1\|1}$	trivial	$\rho(f^0_1)(e^1_1) = \alpha_5 e^1_1$
		$e^1_1 \cdot f^0_1 = \alpha_6 f^1_1$	null

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$\rightsquigarrow$ We have obtained all Lie-Rinehart superalgebras structures on pairs (A, L) with $\dim(A) \leq 2$ and $\dim(L) \leq 4$.

Deformation theory: Super-multiderivations

Let $(A, L[\cdot,\cdot], \rho)$ be a Lie-Rinehart superalgebra and M an A-module.

Definition (Super-multiderivations space)

We define $\mathrm{Der}^n(M, M)$ *as the space of multilinear maps*

$$f : M^{\wedge(n+1)} \longrightarrow M$$

such that it exists an application $\sigma_f : M^{\times n} \longrightarrow \mathrm{Der}(A)$ *(called symbol map), such that*

$$\sigma_f(x_1, \cdots, a \cdot x_i, \cdots, x_n) = (-1)^{|a|(|x_1|+\cdots+|x_{i-1}|)} a \cdot \sigma_f(x_1, \cdots, x_i, \cdots, x_n);$$

$$f(x_1, \cdots, x_n, a \cdot x_{n+1}) = (-1)^{|a|(|f|+|x_1|+\cdots+|x_n|)} a \cdot f(x_1, \cdots, x_{n+1}) + \sigma_f(x_1, \cdots, x_n)(a)(x_{n+1}), \ \forall a \in A.$$

Deformation theory: Super-multiderivations

$$\mathrm{Der}^*(M,M) = \bigoplus_{n\geq -1} \mathrm{Der}^n(M,M), \text{ with } \mathrm{Der}^{-1}(M,M) = M.$$

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Lie structure: $f \in \mathrm{Der}^p(M, M)$ and $g \in \mathrm{Der}^q(M, M)$:

$$[f, g] = f \circ g - (-1)^{pq} g \circ f,$$

$$\text{with symbol map } \sigma_{[f,g]} = \sigma_f \circ g - (-1)^{pq} \sigma_g \circ f + [\sigma_f, \sigma_g],$$

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with

$$(f\circ g)(x_1,\dots,x_{p+q+1}) = \sum_{\tau\in Sh(q+1,p)} \varepsilon(\tau, x_1,\dots,x_{p+q+1})$$
$$\times f\left(g\left(x_{\tau(1)},\dots,x_{\tau(q+1)}\right), x_{\tau(q+2)},\dots,x_{\tau(p+q+1)}\right).$$

Deformation theory: Super-multiderivations

Proposition

There is a one-to-one correspondence between Lie-Rinehart superstructures on (A, L) *and elements* $m \in \mathrm{Der}^1(L, L)$ *such that* $[m, m] = 0$.

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$\rightsquigarrow$ with that in mind, we can construct a deformation theory by identifying Lie-Rinehart superstructures on (A, L) and the corresponding element $m \in \mathrm{Der}^1(L, L)$.

Deformation theory: Deformation cohomology

Cochains space.

$$C^n_{def}(L, L) := \mathrm{Der}^{n-1}(L, L)$$

$$C^*_{def}(L, L) := \bigoplus_{n \geq 0} C^n_{def}(L, L).$$

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We endow it with a differential operator

$$\delta : C^n_{def}(L,L) \longrightarrow C^{n+1}_{def}(L,L),$$
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Proposition

$(C^*_{def}(L, L), \delta)$ *is a cochain complex.*

Deformation theory: main results

Definition

Let $(A, L, [\cdot,\cdot], \rho)$ be a Lie-Rinehart superalgebra, and let $m \in \mathrm{Der}^1(L, L)$ be the corresponding super-multiderivation.

$$\begin{aligned} m_t : L \times L &\longrightarrow L[[t]] \\ (x, y) &\longmapsto \sum_{i \geq 0} t^i m_i(x, y), \quad m_0 = m,\ m_i \in \mathrm{Der}^1(L, L), \end{aligned}$$

Moreover, m_t must verify $[m_t, m_t] = 0$, the bracket being the $\mathbb{Z}$-graded bracket on $C^*_{def}(L[[t]], L[[t]])$.

Deformation theory: main results

Theorem

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Theorem

Any non-trivial deformation of $m \in Der^1(L, L)$ is equivalent to a deformation whose infinitesimal is not a coboundary.

$\Longrightarrow$ If $H^2_{def} = 0$, any deformation is equivalent to the trivial deformation.

The restricted side of the story.

Let $\mathbb{F}$ be a field of prime characteristic p.

Definition (Jacobson)

A **restricted Lie algebra** *is a Lie algebra L endowed with an application $(\cdot)^{[p]} : L \longrightarrow L$ such that*

1. $(\lambda x)^{[p]} = \lambda^p x^{[p]}$, $x \in L$, $\lambda \in \mathbb{F}$;

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3. $(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x,y)$,

with $is_i(x,y)$ the coefficient of Z^{i-1} in $\mathrm{ad}^{p-1}_{Zx+y}(x)$. Such an application $(-)^{[p]} : L \longrightarrow L$ is called a p-map.

Restricted cohomology of restricted Lie algebras

Definition (Restricted 2-cochains; Evans, Fuchs)

Let $\varphi \in C^2_{CE}(L, M)$ *(ordinary Chevalley-Eilenberg 2-cochain) and* $\omega : L \longrightarrow M$*. Then* ω **has the** $(*)$**-property w.r.t** φ *if*

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1. $\omega(\lambda x) = \lambda^p \omega(x), \ \lambda \in \mathbb{F}, \ x \in L;$
2. $\omega(x+y) = \omega(x) + \omega(y) +$
$$\sum_{\substack{x_i = x \text{ or } y \\ x_1 = x, \ x_2 = y}} \frac{1}{\pi(x)} \sum_{k=0}^{p-2} (-1)^k x_p \dots x_{p-k+1} \varphi([[\dots[x_1, x_2], x_3]\dots, x_{p-k-1}], x_{p-k}),$$

with $x, y \in L$, $\pi(x)$ *the number of factors* x_i *equal to* x*.*

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with $x, y \in L$, $\pi(x)$ *the number of factors* x_i *equal to* x*. We then define*

$$C^2_*(L, M) = \Big\{ (\varphi, \omega), \ \varphi \in C^2_{CE}(L, M), \ \omega \text{ has the } (*)\text{-property w.r.t } \varphi \Big\}$$

Restricted cohomology of restricted Lie algebras

- A **restriced 2-cocycle** is an element $(\alpha, \beta) \in C^2_*(L, M)$ such that
  1. α is an ordinary Chevalley-Eilenberg 2-cocycle;
  2. $\alpha\left(x, y^{[p]}\right) - \sum_{i+j=p-1} (-1)^i y^i \alpha\left([x, \underbrace{y, ..., y}_{j \text{ terms}}], y\right) + x\beta(y) = 0.$

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- A **restriced 2-coboundary** is an element $(\alpha, \beta) \in C^2_*(L, M)$ such that $\exists \varphi \in \mathrm{Hom}(L, M)$,
  1. $\alpha(x, y) = \varphi([x, y]) - x\varphi(y) + y\varphi(x)$;
  2. $\beta(x) = \varphi\left(x^{[p]}\right) - x^{p-1}\varphi(x).$

We are in the following situation:

$$0 \longrightarrow C^0_*(L,M) \xrightarrow{d^0_*} C^1_*(L,M) \xrightarrow{d^1_*} C^2_*(L,M) \xrightarrow{d^2_*} C^3_*(L,M)$$

with $d^0_* = d^0_{CE}$.

Restricted Formal Deformations

Definition

A restricted formal deformation of L is given by:

$$[\cdot,\cdot]_t : (x,y) \longmapsto \sum_{i\geq 0} t^i \mu_i(x,y), \quad (\cdot)^{[p]_t} : x \longmapsto \sum_{j\geq 0} t^j \omega_j(x),$$

with $\mu_0(x,y) = [x,y]$, μ_i *skewsymmetric,* $\omega_0 = (\cdot)^{[p]}$, ω_j *such that* $\omega_j(\lambda x) = \lambda^p \omega_j(x)$.
Moreover, $[\cdot,\cdot]_t$ *et* $(\cdot)^{[p]_t}$ *must satisfy*

$$[x,[y,z]_t]_t + [y,[z,x]_t]_t + [z,[x,y]_t]_t = 0; \qquad (1)$$

$$\left[x, y^{[p]_t}\right]_t = [[\ldots[x,\overbrace{y]_t,y]_t,\ldots,y}^{p\ \text{terms}}]_t. \qquad (2)$$

Restricted Formal Deformations

Proposition

Let $\left([\cdot,\cdot]_t,(\cdot)^{[p]_t}\right)$ *be a restricted deformation of* $\left(L,[\cdot,\cdot],(\cdot)^{[p]}\right)$. *Then* (μ_1,ω_1) *is a 2-cocyle of the restricted cohomology.*

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Proposition

Let $\left([\cdot,\cdot]_t^1,(\cdot)^{[p]_t^1}\right)$ *and* $\left([\cdot,\cdot]_t^2,(\cdot)^{[p]_t^2}\right)$ *be two equivalent formal deformations of* $\left(L,[\cdot,\cdot],(\cdot)^{[p]}\right)$. *Then, their infinitesimal elements are in the same cohomological class.*

The restricted side of the story.

Let A be an associative commutative algebra, $D \in \mathrm{Der}(A)$ and $a \in A$. In characteristic p, we have:

$$(aD)^p = a^p D^p + (aD)^{p-1}(a)D.$$

Restricted Lie-Rinehart Algebras.

Definition

Let A be an associative commutative algebra over a field $\mathbb{F}$ of characteristic p. Then (A, L) is a **restricted Lie-Rinehart algebra** *if*

1. *(A, L) is a Lie-Rinehart algebra, with anchor map* $\rho : L \longrightarrow \mathrm{Der}(A)$;
2. $\left(L, (\cdot)^{[p]}\right)$ *is a restricted Lie algebra;*
3. $\rho(x^{[p]}) = \rho(x)^p$;
4. $(ax)^{[p]} = a^p x^{[p]} + \rho(ax)^{p-1}(a)x, \ a \in A, \ x \in L.$

Definition

A **restricted multiderivation** *(of order 1) is a pair* (m, ω)*, where* $m : L \times L \to L$ *is skew-symmetric,* ω *is p-homogeneous and satisfies*

$$\omega(x+y) = \omega(x) + \omega(y) + \sum_{i=1}^{p-1} \theta_i(x,y), \tag{3}$$

where $i\theta_i(x,y)$ *is the coefficient of* Z^{i-1} *in* $\left(\tilde{\mathrm{ad}}_m(Zx+y)\right)^{p-1}(x)$*, with* $\tilde{\mathrm{ad}}_m(x)(y) := m(x,y)$*, such that it exists a map* $\sigma_m : L \to \mathrm{Der}(A)$ *called* **restricted symbol map** *which must satisfy the following four conditions, for* $x, y \in L$ *and* $a \in A$*:*

$$\sigma(ax) = a\sigma(x); \tag{4}$$

$$m(x, ay) = am(x,y) + \sigma(x)(a)y; \tag{5}$$

$$\sigma \circ \omega(x) = \sigma(x)^p; \tag{6}$$

$$\omega(ax) = a^p\,\omega(x) + \sigma(ax)^{p-1}(a)x. \tag{7}$$

Proposition

There is a one-to-one correspondence between restricted Lie-Rinehart structures on the pair (A, L) and restricted multiderivations of order 1 such that

$$m(x, m(y, z)) + m(y, m(z, x)) + m(z, m(x, y)) = 0 \qquad (8)$$

and

$$m(x, \omega(y)) = m(m(\ldots m(x, \overbrace{y), y), \ldots, y}^{p \text{ terms}}) \qquad (9)$$

Thank you for your attention.