Deformations of Lie-Rinehart (Super)algebras

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1. Basics
2. Deformations of Lie-Rinehart superalgebras in char $= 0$.
3. The restricted side of the story: positive characteristic.

Definitions

$\mathbb{K}$ is a field of characteristic 0.

Definition

A **Lie superalgebra** *L is a $\mathbb{Z}_2$-graded $\mathbb{K}$-module $L = L_0 \bigoplus L_1$ endowed with a bracket $[\cdot,\cdot]$ (called super Lie bracket) satisfying, for homogeneous elements $x, y, z \in L$:*

- $|[x,y]| = |x| + |y|$;
- $[x,y] = -(-1)^{|x||y|}[y,x]$ *(super-skewsymmetry);*
- $(-1)^{|x||z|}[x,[y,z]] + (-1)^{|z||y|}[z,[x,y]] + (-1)^{|x||y|}[y,[z,x]] = 0$ *(super-Jacobi).*

Definitions

Definition

An **associative superalgebra** *is a* $\mathbb{Z}_2$*-graded* $\mathbb{K}$*-module* $A = A_0 \oplus A_1$ *endowed with a bilinear map* $A \times A \longrightarrow A$ *such that* $(ab)c = a(bc)$ *for all* $a, b, c \in A$*, and* $A_i A_j \subset A_{i+j}$.

It is called **supercommutative** *if* $ab = (-1)^{|a||b|} ba, \ \forall a, b$ *homogeneous in* A.

Definitions

Definition

Let A be a superalgebra. A map $D : A \longrightarrow A$ is a **superderivation** *(of degree $|D|$) of A if D is a $\mathbb{Z}_2$-graded linear map and if the super-Leibniz condition is satisfied:*

$$D(ab) = D(a)b + (-1)^{|a||D|} aD(b) \quad \forall a, b \in A.$$

We denote by $\mathrm{Der}(A)$ *the vector superspace of superderivations of A.*

Definitions

Definition

A **Lie-Rinehart superalgebra** *is a pair* (A, L), *where*

- L *is a Lie* $\mathbb{K}$*-superalgebra, endowed with a bracket* $[\cdot,\cdot]$;
- A *is an associative and supercommutative* $\mathbb{K}$*-superalgebra,*

such that, for $x, y \in L$ *and* $a, b \in A$:

- *There is an action* $A \times L \longrightarrow L, \ (a, x) \longmapsto a \cdot x$, *making* L *an* A*-module;*
- *There is a map* $\rho : L \longrightarrow Der(A), \ x \longmapsto \rho_x$, *which is both a Lie superalgebra morphism and a* A*-module morphism, called* **anchor map**;
- $[x, a \cdot y] = \rho_x(a) \cdot y + (-1)^{|a||x|} a \cdot [x, y]$.

An example

Lie superalgebra of superderivations. Let A be an associative supercommutative unital superalgebra, and $L = \mathrm{Der}(A)$ be its superderivations superalgebra. Then $(A, \mathrm{Der}(A))$ is a Lie-Rinehart superalgebra, with $\rho = \mathrm{id}$.

Deformation theory: Super-multiderivations

Let $(A, L[\cdot,\cdot], \rho)$ be a Lie-Rinehart superalgebra and M an A-module.

Definition (Super-multiderivations space)

We define $\mathrm{Der}^n(M, M)$ *as the space of multilinear maps*

$$f : M^{\wedge(n+1)} \longrightarrow M$$

such that it exists an application $\sigma_f : M^{\times n} \longrightarrow \mathrm{Der}(A)$ *(called symbol map), such that*

$$\sigma_f(x_1, \cdots, a \cdot x_i, \cdots, x_n) = (-1)^{|a|(|x_1|+\cdots+|x_{i-1}|)} a \cdot \sigma_f(x_1, \cdots, x_i, \cdots, x_n);$$

$$f(x_1, \cdots, x_n, a \cdot x_{n+1}) = (-1)^{|a|(|f|+|x_1|+\cdots+|x_n|)} a \cdot f(x_1, \cdots, x_{n+1}) + \sigma_f(x_1, \cdots, x_n)(a)(x_{n+1}), \ \forall a \in A.$$

Deformation theory: Super-multiderivations

$$\mathrm{Der}^*(M,M)=\bigoplus_{n\geq -1}\mathrm{Der}^n(M,M),\ \text{ with }\mathrm{Der}^{-1}(M,M)=M.$$

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Lie structure: $f \in \mathrm{Der}^p(M,M)$ and $g \in \mathrm{Der}^q(M,M)$:

$$[f,g] = f \circ g - (-1)^{pq} g \circ f,$$

$$\text{with symbol map } \sigma_{[f,g]} = \sigma_f \circ g - (-1)^{pq} \sigma_g \circ f + [\sigma_f, \sigma_g],$$

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with

$$(f\circ g)(x_1,\dots,x_{p+q+1}) = \sum_{\tau\in Sh(q+1,p)} \varepsilon(\tau, x_1,\dots,x_{p+q+1})$$
$$\times f\left(g(x_{\tau(1)},\dots,x_{\tau(q+1)}), x_{\tau(q+2)},\dots,x_{\tau(p+q+1)}\right).$$

Deformation theory: Super-multiderivations

Proposition

There is a one-to-one correspondence between Lie-Rinehart superstructures on (A, L) *and elements* $m \in \mathrm{Der}^1(L, L)$ *such that* $[m, m] = 0$.

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$\rightsquigarrow$ with that in mind, we can construct a deformation theory by identifying Lie-Rinehart superstructures on (A, L) and the corresponding element $m \in \mathrm{Der}^1(L, L)$.

Deformation theory: Deformation cohomology

Cochains space.

$$C^n_{def}(L, L) := \mathrm{Der}^{n-1}(L, L)$$
$$C^*_{def}(L, L) := \bigoplus_{n\geq 0} C^n_{def}(L, L).$$

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We endow it with a differential operator

$$\delta : C^n_{def}(L,L) \longrightarrow C^{n+1}_{def}(L,L),$$
$$D \longmapsto [m,D].$$

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Proposition

$(C^*_{def}(L,L),\delta)$ *is a cochain complex.*

Deformation theory: main results

Definition

Let $(A, L, [\cdot,\cdot], \rho)$ be a Lie-Rinehart superalgebra, and let $m \in \mathrm{Der}^1(L, L)$ be the corresponding super-multiderivation.

$$\begin{aligned} m_t : L \times L &\longrightarrow L[[t]] \\ (x,y) &\longmapsto \sum_{i\geq 0} t^i m_i(x,y), \quad m_0 = m,\ m_i \in \mathrm{Der}^1(L,L), \end{aligned}$$

Moreover, m_t must verify $[m_t, m_t] = 0$, the bracket being the $\mathbb{Z}$-graded bracket on $C^*_{def}(L[[t]], L[[t]])$.

Deformation theory: main results

Theorem

Let m_t be a deformation of a Lie-Rinehart superalgebra (A, L). Then the infinitesimal m_1 is a 2-cocycle with respect to the deformation cohomology.

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Theorem

Any non-trivial deformation of $m \in Der^1(L, L)$ is equivalent to a deformation whose infinitesimal is not a coboundary.

$\Longrightarrow$ If $H^2_{def} = 0$, any deformation is equivalent to the trivial deformation.

The restricted side of the story.

Let $\mathbb{F}$ be a field of prime characteristic $p > 2$.

Definition (Jacobson)

A **restricted Lie algebra** *is a Lie algebra L endowed with an application $(\cdot)^{[p]} : L \longrightarrow L$ such that*

1. $(\lambda x)^{[p]} = \lambda^p x^{[p]}$, $x \in L$, $\lambda \in \mathbb{F}$;

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2. $\left[x, y^{[p]}\right] = [[...[x, \overbrace{y], y], ..., y}^{p \text{ terms}}]$;
3. $(x + y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x, y)$,

with $is_i(x, y)$ *the coefficient of* Z^{i-1} *in* $\mathrm{ad}^{p-1}_{Zx+y}(x)$. *Such an application* $(-)^{[p]} : L \longrightarrow L$ *is called a p-map.*

An example: the Witt algebra $W(1)$.
Let char $(\mathbb{F}) = p \geq 5$. We define

$$W(1) = \text{Span}\{e_{-1}, e_0, \dots, e_{p-2}\}$$

endowed with the bracket

$$[e_i, e_j] = \begin{cases} (j-i)e_{i+j} & \text{if} \quad i+j \in \{-1, \dots, p-2\}; \\ 0 \quad \text{otherwise}; \end{cases}$$

and the p-map

$$e_i^{[p]} = \begin{cases} e_0^{[p]} = e_0; \\ e_i^{[p]} = 0 \quad \text{if } i \neq 0. \end{cases}$$

Restricted cohomology of restricted Lie algebras

Definition (Restricted 2-cochains; Evans, Fuchs)

Let $\varphi \in C^2_{CE}(L, M)$ *(ordinary Chevalley-Eilenberg 2-cochain) and* $\omega : L \longrightarrow M$. *Then* ω **has the** $(*)$**-property w.r.t** φ *if*

1. $\omega(\lambda x) = \lambda^p \omega(x), \ \lambda \in \mathbb{F}, \ x \in L;$

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1. $\omega(\lambda x) = \lambda^p \omega(x), \ \lambda \in \mathbb{F}, \ x \in L;$
2. $\omega(x+y) = \omega(x) + \omega(y) +$
$$\sum_{\substack{x_i = x \text{ or } y \\ x_1 = x, \ x_2 = y}} \frac{1}{\pi(x)} \sum_{k=0}^{p-2} (-1)^k x_p \ldots x_{p-k+1} \varphi([[\ldots[x_1, x_2], x_3] \ldots, x_{p-k-1}], x_{p-k}),$$

with $x, y \in L$, $\pi(x)$ the number of factors x_i equal to x.

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with $x, y \in L$, $\pi(x)$ the number of factors x_i equal to x.

$$C^2_*(L, M) = \Big\{ (\varphi, \omega), \ \varphi \in C^2_{CE}(L, M), \ \omega \text{ has the } (*)\text{-property w.r.t } \varphi \Big\}$$

Restricted cohomology of restricted Lie algebras

- A **restricted 2-cocycle** is an element $(\alpha,\beta)\in C^2_*(L,M)$ such that
  1. α is an ordinary Chevalley-Eilenberg 2-cocycle;
  2. $\alpha\left(x,y^{[p]}\right)-\sum_{i+j=p-1}(-1)^i y^i\alpha\left([x,\underbrace{y,\dots,y}_{j\text{ terms}}],y\right)+\ x\beta(y)=0.$

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  1. α is an ordinary Chevalley-Eilenberg 2-cocycle;
  2. $\alpha\left(x, y^{[p]}\right) - \sum_{i+j=p-1} (-1)^i y^i \alpha\left([x, \underbrace{y, ..., y}_{j \text{ terms}}], y\right) + x\beta(y) = 0.$
- A **restricted 2-coboundary** is an element $(\alpha, \beta) \in C^2_*(L, M)$ such that $\exists \varphi \in \mathrm{Hom}(L, M)$,
  1. $\alpha(x, y) = \varphi([x, y]) - x\varphi(y) + y\varphi(x);$
  2. $\beta(x) = \varphi\left(x^{[p]}\right) - x^{p-1}\varphi(x).$

We are in the following situation:

$$0 \longrightarrow C_*^0(L,M) \xrightarrow{d_*^0} C_*^1(L,M) \xrightarrow{d_*^1} C_*^2(L,M) \xrightarrow{d_*^2} C_*^3(L,M)$$

with $d_*^0 = d_{CE}^0$.

The restricted side of the story.

Let A be an associative commutative algebra.

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- $(A, \mathrm{Der}(A), \mathrm{id})$ is an ordinary Lie-Rinehart algebra;
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Let A be an associative commutative algebra.

- $(A, \mathrm{Der}(A), \mathrm{id})$ is an ordinary Lie-Rinehart algebra;
- $(\mathrm{Der}(A), (\cdot)^p)$ is a restricted Lie algebra;
- If $D \in \mathrm{Der}(A)$ and $a \in A$, we have

$$(aD)^p = a^p D^p + (aD)^{p-1}(a)D.$$

Restricted Lie-Rinehart Algebras.

Definition

Let A be an associative commutative algebra over a field $\mathbb{F}$ of characteristic p. Then (A, L) is a **restricted Lie-Rinehart algebra** *if*

1. *(A, L) is a Lie-Rinehart algebra, with anchor map* $\rho : L \longrightarrow \mathrm{Der}(A)$;
2. $\left(L, (\cdot)^{[p]}\right)$ *is a restricted Lie algebra;*
3. $\rho(x^{[p]}) = \rho(x)^p$;
4. $(ax)^{[p]} = a^p x^{[p]} + \rho(ax)^{p-1}(a)x, \;\; a \in A, \;\; x \in L.$

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1. *(A, L) is a Lie-Rinehart algebra, with anchor map $\rho : L \longrightarrow \mathrm{Der}(A)$;*
2. *$\left(L, (\cdot)^{[p]}\right)$ is a restricted Lie algebra;*
3. *$\rho(x^{[p]}) = \rho(x)^p$;*
4. *$(ax)^{[p]} = a^p x^{[p]} + \rho(ax)^{p-1}(a)x, \; a \in A, \; x \in L.$*

Example: The Witt algebra with $A = \mathbb{F}[x]/(x^p - 1)$.

Definition

A **restricted multiderivation** *(of order 1) is a pair (m, ω), where $m : L \times L \to L$ is skew-symmetric, ω is p-homogeneous and satisfies*

$$\omega(x+y) = \omega(x) + \omega(y) + \sum_{i=1}^{p-1} \theta_i(x,y), \tag{1}$$

where $i\theta_i(x,y)$ is the coefficient of Z^{i-1} in $\left(\tilde{\mathrm{ad}}_m(Zx+y)\right)^{p-1}(x)$, with $\tilde{\mathrm{ad}}_m(x)(y) := m(x,y)$, such that it exists a map $\sigma_m : L \to \mathrm{Der}(A)$ called **restricted symbol map** *which must satisfy the following four conditions, for $x, y \in L$ and $a \in A$:*

$$\sigma(ax) = a\sigma(x); \tag{2}$$

$$m(x, ay) = am(x,y) + \sigma(x)(a)y; \tag{3}$$

$$\sigma \circ \omega(x) = \sigma(x)^p; \tag{4}$$

$$\omega(ax) = a^p \omega(x) + \sigma(ax)^{p-1}(a)x. \tag{5}$$

Proposition

There is a one-to-one correspondence between restricted Lie-Rinehart structures on the pair (A, L) and restricted multiderivations of order 1 such that

$$m(x, m(y, z)) + m(y, m(z, x)) + m(z, m(x, y)) = 0 \tag{6}$$

and

$$m(x, \omega(y)) = m(m(\ldots m(x, \overbrace{y), y), \ldots, y}^{p \text{ terms}})) \tag{7}$$

Restricted Formal Deformations

Definition

A formal deformation of (m, ω) *is given, for* $x, y \in L$*, by two applications*

$$m_t : (x, y) \longmapsto \sum_{i \geq 0} t^i m_i(x, y), \quad \omega_t : x \longmapsto \sum_{j \geq 0} t^j \omega_j(x),$$

with $m_0 = m$*,* $\omega_0 = \omega$*, and* (m_i, ω_i) *restricted multiderivations. Moreover, the four following conditions must be satisfied, for* $x, y, z \in L$*, and* $a \in A$*:*

$$m_t(x, m_t(y, z))_t + m_t(y, m_t(z, x)) + m_t(z, m_t(x, y)) = 0; \tag{8}$$

$$m_t\left(x, \omega_t(y)\right)_t = m_t(m_t(\cdots m_t(x, \overbrace{y), y), \cdots, y}^{p \text{ terms}}); \tag{9}$$

$$\sum_{i=0}^{k} \sigma_i\left(\omega_{k-i}(x)\right)(a) = \sum_{i_1+\cdots+i_p=k} \sigma_{i_1}(x) \circ \cdots \circ \sigma_{i_p}(x)(a), \quad \forall k \geq 0; \tag{10}$$

$$\sigma_k(x)^{p-1} = \sum_{i_1+\cdots+i_{p-1}=k} \sigma_{i_1}(x) \circ \sigma_{i_2}(x) \circ \cdots \sigma_{i_{p-1}}(x) \quad \forall k \geq 0; \tag{11}$$

Restricted Formal Deformations

Proposition

Let (m_t, ω_t) be a restricted deformation of (m, ω). Then (m_1, ω_1) is a 2-cocyle of the restricted cohomology.

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Proposition

Let (m_t, ω_t) and (m'_t, ω'_t) be two equivalent formal deformations of (m, ω). Then, their infinitesimal elements are in the same cohomological class.

Thank you for your attention.

Classification: an example

(1|1, 1|1)-type: $(\alpha_i \in \mathbb{C},\ \alpha_i \neq 0)$

A	L	Action	Anchor
$A^1_{1\|1}$	$L^1_{1\|1}$	$e^1_1 \cdot f^1_1 = \alpha_1 f^0_1$	null
	$L^2_{1\|1}$	trivial	$\rho(f^0_1)(e^1_1) = \alpha_2 e^1_1$
		$e^1_1 \cdot f^1_1 = \alpha_3 f^0_1$	$\rho(f^0_1)(e^1_1) = -e^1_1,\ \rho(f^1_1)(e^1_1) = -\alpha_3 e^0_1$
		$e^1_1 \cdot f^0_1 = \alpha_4 f^1_1$	$\rho(f^0_1)(e^1_1) = e^1_1$
	$L^3_{1\|1}$	trivial	$\rho(f^0_1)(e^1_1) = \alpha_5 e^1_1$
		$e^1_1 \cdot f^0_1 = \alpha_6 f^1_1$	null

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	$L^2_{1\|1}$	trivial	$\rho(f^0_1)(e^1_1) = \alpha_2 e^1_1$
		$e^1_1 \cdot f^1_1 = \alpha_3 f^0_1$	$\rho(f^0_1)(e^1_1) = -e^1_1,\ \rho(f^1_1)(e^1_1) = -\alpha_3 e^0_1$
		$e^1_1 \cdot f^0_1 = \alpha_4 f^1_1$	$\rho(f^0_1)(e^1_1) = e^1_1$
	$L^3_{1\|1}$	trivial	$\rho(f^0_1)(e^1_1) = \alpha_5 e^1_1$
		$e^1_1 \cdot f^0_1 = \alpha_6 f^1_1$	null

$\rightsquigarrow$ We have obtained all Lie-Rinehart superalgebras structures on pairs (A, L) with $\dim(A) \leq 2$ and $\dim(L) \leq 4$.