Chevalley-Eilenberg cohomology and deformations of Lie algebras

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Introduction

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Example:

$$\mathbf{SL}_2(\mathbb{C}) \rightleftarrows \mathfrak{sl}_2(\mathbb{C})$$

$$\mathbf{SL}_2(\mathbb{C}) = \{M \in GL_n(\mathbb{C}),\ \det(M) = 1\}$$

$$\mathfrak{sl}_2(\mathbb{C}) = \{M \in M_n(\mathbb{C}),\ Tr(M) = 0\}$$

Characteristic 0 - Basics

Let $\mathbb{K}$ be an algebraic closed field of characteristic 0.

Definition

Let L be a $\mathbb{K}$ vector space. A **Lie bracket** *on L is a bilinear map $[\cdot,\cdot] : L \times L \longrightarrow L$ satisfying, for $x, y, z \in L$,*

1. $[x, y] = -[y, x]$ *(anticommutativity)*
2. $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ *(Jacobi identity).*

If L is endowed with such a bracket, we call the pair $(L, [\cdot,\cdot])$ a **Lie algebra**.

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- Let $\mu : L \times L \longrightarrow L$ be an associative multiplication. Then

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is a Lie bracket on L called **commutator**.

Important consequence: $M_n(\mathbb{K})$ endowed with the commutator is a Lie algebra.

Characteristic 0 - Basics

Definition

A linear map $\varphi : L_1 \longrightarrow L_2$ *is a* **Lie algebra map** *if*

$$\varphi([x,y]_1) = [\varphi(x), \varphi(y)]_2 \ \forall \ x, y \in L_1.$$

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Definition

A representation $\rho : L \longrightarrow End(V)$ *is a* **Lie algebra representation** *if* ρ *is a Lie algebra map, that is,*

$$\rho([x, y]) = \rho(x)\rho(y) - \rho(y)\rho(x).$$

Remark: One can also say that V is a L-module.

Characteristic 0 - Basics

Crucial example: the adjoint representation

$$\begin{aligned} \mathrm{ad} : L &\longrightarrow \mathrm{End}(L) \\ x &\longmapsto \mathrm{ad}_x : y \longmapsto [x, y]. \end{aligned}$$

Characteristic 0 - Chevalley-Eilenberg cohomology

We aim to construct a **cochain complex** associated to L:

$$0 \longrightarrow C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^2 \xrightarrow{d^2} C^3 \xrightarrow{d^3} \dots$$

The C^i being L-modules and the d^j linear maps satisfying

$$d^{j+1} \circ d^j = 0.$$

Characteristic 0 - Chevalley-Eilenberg cohomology

Let L be a Lie algebra and M a L-module. For $q \geq 0$, we set:

$$C^q_{CE}(L, M) := \mathrm{Hom}_{\mathbb{K}}\left(\wedge^q L, M\right).$$

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They satisfy

$$\forall \sigma \in \mathfrak{S}_q, \ \varphi(x_{\sigma(1)}, ..., x_{\sigma(q)}) = \mathrm{sign}(\sigma)\varphi(x_1, ..., x_q).$$

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Remark: if $(L, [\cdot,\cdot])$ is a Lie algebra, then the bracket $[\cdot,\cdot]$ belongs to $C^2_{CE}(L, L)$.

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Now, we need to consider **differential maps**

$$d^q_{CE} : C^q_{CE}(L, M) \longrightarrow C^{q+1}_{CE}(L, M),$$

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given by

$$d^q_{CE}\varphi(x_1, \ldots, x_{q+1}) =$$

$$\sum_{1 \leq i < j \leq q+1} (-1)^{i+j-1} \varphi\left([x_i, x_j], x_1, \ldots \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{q+1}\right)$$
$$+ \sum_{1 \leq i}^{q+1} (-1)^i x_i \cdot \varphi\left(x_1, \ldots, \hat{x}_i, \ldots, x_{q+1}\right).$$

Characteristic 0 - Chevalley-Eilenberg cohomology

Examples of differential maps.

$$d^q_{CE}\varphi(x_1, ..., x_{q+1}) = \sum_{1\leq i<j\leq q+1} (-1)^{i+j-1}\varphi\left([x_i, x_j], x_1, ...\hat{x}_i, ..., \hat{x}_j, ..., x_{q+1}\right) + \sum_{1\leq i}^{q+1}(-1)^i x_i \cdot \varphi\left(x_1, ..., \hat{x}_i, ..., x_{q+1}\right).$$

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$$d^1\varphi(x_1,x_2) = \varphi([x_1,x_2]) - x_1\cdot\varphi(x_2) + x_2\cdot\varphi(x_1).$$

$$d^2\psi(x_1,x_2,x_3) = \psi([x_1,x_2],x_3) - \psi([x_1,x_3],x_2) + \psi([x_2,x_3],x_1) - x_1\cdot\psi(x_2,x_3) + x_2\cdot\psi(x_1,x_3) - x_3\cdot\psi(x_1,x_2).$$

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Lemma

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4. We have $B_{CE}^q(L, M) \subset Z_{CE}^q(L, M)$. We can consider the quotient space

$$H_{CE}^q(L, M) := Z_{CE}^q(L, M)/B_{CE}^q(L, M).$$

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Then,

$$Z^1_{CE}(L,L) = \left\{\varphi \in C^1_{CE}(L,L),\ \varphi([x_1,x_2]) = [x_1,\varphi(x_2)] + [\varphi(x_1),x_2]\right\}.$$

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Finally,

$$H^1_{CE}(L,L) = \{\text{derivations of } L\} \,/\, \{\text{inner derivations of } L\} = \ \mathrm{Out}(L).$$

Characteristic 0 - Deformations

Definition

Let $(L, [\cdot,\cdot])$ *a Lie algebra. A* **formal deformation** *of* L *is a* $\mathbb{K}[[t]]$*-bilinear map*

$$\mu_t : L[[t]] \times L[[t]] \longrightarrow L[[t]],$$

defined on $L \times L$ *by*

$$\mu_t(x,y) = [x,y] + \sum_{i\geq 1} t^i \mu_i(x,y),$$

with $\mu_i : L \times L \longrightarrow L$ *bilinear skewsymmetric satisfying*

$$\mu_t(x, \mu_t(y,z)) + \mu_t(y, \mu_t(z,x)) + \mu_t(z, \mu_t(x,y)) = 0, \ \forall x,y,z \in L.$$

Characteristic 0 - Deformations

$$\mu_t(x,\mu_t(y,z))+\mu_t(y,\mu_t(z,x))+\mu_t(z,\mu_t(x,y))=0,\ \forall x,y,z\in L.$$

We expand this equation and we get (for every $q\geq 0$)

$$\sum_{i=0}^{q}\left(\mu_i(x,\mu_{q-i}(y,z))+\mu_i(y,\mu_{q-i}(z,x))+\mu_i(z,\mu_{q-i}(x,y))\right)=0$$

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- Taking $q=0$, we recover the Jacobi identity;
- Taking $q=1$, we obtain $\mu_1\in Z^2_{CE}(L,L)$.

Characteristic 0 - Deformations

Obstructions.

The study of the converse of this last result leads to the theory of obstructions.

Definition

Let $\varphi \in Z^2_{CE}(L, L)$ be a 2-cocycle. Then φ is called **integrable** *if there is a formal deformation μ_t of L such that $\mu_1 = \varphi$.*

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Definition

A n-order deformation of L is a deformation of the form

$$\mu_t^n = \sum_{i=0}^{n} t^i \mu_i .$$

Characteristic 0 - Deformations

Definition

Let μ_t^n be a n-order deformation of L. We define for $x, y, z \in L$:

$$obs_n(x,y,z) = \sum_{i=1}^{n} \mu_i(x, \mu_{n+1-i}(y,z)) + \mu_i(y, \mu_{n+1-i}(z,x)) + \mu_i(z, \mu_{n+1-i}(x,y)).$$

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Proposition

Let μ_t^n be a n-order deformation of L. Then $obs_n \in Z^3_{CE}(L,L)$.

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Proposition

Let μ_t^n be a n-order deformation of L. We set

$$\mu_t^{n+1} = \mu_t^n + t^{n+1}\mu_{n+1},$$

for $\mu_{n+1} \in C^2_{CE}(L, L)$. Then

$$\mu_t^{n+1} \text{ is a } n+1\text{-order deformation of } L \iff obs_n = d^2_{CE}\mu_{n+1}.$$

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Theorem

1. *If $H^3_{CE}(L, L) = 0$, every 2-cocycle is integrable.*
2. *A n-order deformation of L extends to a $n+1$-order deformation if and only if the cohomology class of obs_n vanishes.*

Characteristic 0 - Deformations

Equivalence of formal deformations.

Let V be a vector space, a **formal automorphism** $\phi_t : V[[t]] \longrightarrow V[[t]]$ is given by a family of linear maps $\phi_i : L \longrightarrow L$ satisfying $\phi_t = \sum_{i \geq 0} t^i \phi_i$, with $\phi_0 = id$.

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Definition

Let μ_t and ν_t be two formal deformations of a Lie algebra L. They are **equivalent** *if there is a formal automorphism ϕ_t such that, for $x, y \in L$,*

$$\phi_t(\mu_t(x, y)) = \nu_t(\phi_t(x), \phi_t(y)).$$

Characteristic 0 - Deformations

Theorem

1. *Every non-trivial deformation of L is equivalent to a deformation of the form*

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2. *If $H^2_{CE}(L,L) = 0$, every deformation of L is trivial.*

Remark: there is a one-to-one correspondence between elements of $H^2_{CE}(L,L)$ and infinitesimal elements of non-equivalent deformations. That is, $H^2_{CE}(L,L)$ fully classifies the infinitesimal deformations of the form $\mu_t = [\cdot,\cdot] + t\mu_1$.

Positive characteristic - restricted Lie algebras

Let $\mathbb{F}$ a field of characteristic $p > 2$ and A an associative $\mathbb{F}$-algebra. With the commutator, it's a Lie algebra. The adjoint representation is then given by

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Then, if $m = p$, we obtain

$$\mathrm{ad}_x^p(y) = x^p y - y x^p = \mathrm{ad}_{x^p}(y).$$

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Lemma

Let A be an associative algebra and let $a, b \in A$. Then,

$$(a+b)^p = a^p + b^p + \sum_{i=1}^{p-1} s_i(a,b),$$

with $is_i(a,b)$ being the coefficient of X^{i-1} in the polynomial expression $\mathrm{ad}_{aX+b}^{p-1}(a)$.

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$\rightsquigarrow$ it's much less friendly.

Positive characteristic - the p-mappings

The following definition is motivated by the previous example.

Definition (Jacobson)

A **restricted Lie algebra** *is a Lie algebra* L *equipped with a map* $(\cdot)^{[p]} : L \longrightarrow L$ *satisfying*

1. $(\lambda x)^{[p]} = \lambda^p x^{[p]}$, $x \in L$, $\lambda \in \mathbb{F}$;

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3. $(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x,y)$,

with $is_i(x,y)$ the coefficient of Z^{i-1} in $\mathrm{ad}^{p-1}_{Zx+y}(x)$. Such a map $(-)^{[p]} : L \longrightarrow L$ is called p-map.

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1. Every associative algebra can be endowed with a restricted Lie algebra structure with the Frobenius map $x \longmapsto x^p$.

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2. If L is abelian, any p-semilinear map

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is a p-map.

3. Explicit expression for the sum of the s_i:

$$\sum_{i=1}^{p-1} s_i(x,y) = \sum_{\substack{x_i = x \text{ or } y \\ x_p = x, \ x_{p-1} = y}} \frac{1}{\sharp\{x\}} [x_1, [x_2, [\dots, [x_{p-1}, x_p] \dots],$$

Positive characteristic - the p-mappings

Examples:

1. Let A be an associative algebra and

$$\mathrm{Der}(A) = \{D : A \to A \text{ linear}, \ D(ab) = D(a)b + aD(b) \ \forall a, b \in A\}.$$

Then, $\mathrm{Der}(A)$ is a restricted Lie algebra with the commutator and the p-mapping $D \mapsto D^p$.

Positive characteristic - the p-mappings

Examples:

2. Restricted $\mathfrak{sl}_2(\mathbb{F})$ (char $\mathbb{F} > 2$):

$$\mathfrak{sl}_2(\mathbb{F}) = \text{span}_{\mathbb{F}} \{X, Y, H\},$$

with brackets $[X, Y] = H, \ [H, X] = 2X, \ [H, Y] = -2Y$.

Positive characteristic - the p-mappings

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with brackets $[X, Y] = H, \ [H, X] = 2X, \ [H, Y] = -2Y$.
Then,

$$X^{[p]} = Y^{[p]} = 0, \ H^{[p]} = 2^{p-1} H$$

is the **unique** p-structure on $\mathfrak{sl}_2(\mathbb{F})$.

Positive characteristic - the restricted cohomology

Let L be a restricted Lie algebra and M a L-module.

We aim to define the **restricted cohomology** of L, denoted $C^q_*(L, M)$, $q \geq 0$. We set:

$$C^0_*(L, M) = C^0_{CE}(L, M); \quad C^1_*(L, M) = C^1_{CE}(L, M).$$

Positive characteristic - the restricted cohomology

Definition

Let $\varphi \in C^2_{CE}(L, M)$ et $\omega : L \longrightarrow M$. We say that ω **has the $(*)$-property w.r.t.** φ *if*

1. $\omega(\lambda x) = \lambda^p \omega(x), \ \lambda \in \mathbb{F}, \ x \in L;$
2. $\omega(x+y) = \omega(x) + \omega(y) +$

$$\sum_{\substack{x_i = x \text{ or } y \\ x_1 = x, \ x_2 = y}} \frac{1}{\pi(x)} \sum_{k=0}^{p-2} (-1)^k x_p \dots x_{p-k+1} \varphi([[\dots[x_1, x_2], x_3] \dots, x_{p-k-1}], x_{p-k}),$$

with $x, y \in L$, $\pi(x)$ the number of x_i equal to x. We then define

$$C^2_*(L, M) = \left\{ (\varphi, \omega), \ \varphi \in C^2_{CE}(L, M), \ \omega \text{ has the } (*)\text{-property w.r.t. } \varphi \right\}.$$

Positive characteristic - the restricted cohomology

Definition

Let $\alpha \in C^3_{CE}(L, M)$ *et* $\beta : L \times L \longrightarrow M$. *We say that* β **has the** $(**)$**-property w.r.t** α *if*

1. $\beta(\cdot, y)$ *is linear;*
2. $\beta(x, \lambda y) = \lambda^p \beta(x, y)$*;*
3. $\beta(x, y_1 + y_2) = \beta(x, y_1) + \beta(x, y_2) -$

$$\sum_{\substack{h_i = y_1 \text{ or } y_2 \\ h_1 = y_1,\ h_2 = y_2}} \frac{1}{\pi(y_1)} \sum_{j=0}^{p-2} (-1)^j \sum_{k=1}^{j} \binom{j}{k} h_p \dots h_{p-k-1} \alpha\left([x, h_{p-k}, \dots, h_{p-j+1}], [h_1, \dots, h_{p-j-1}], h_{p-j}\right)$$

with $\lambda \in \mathbb{F}$, $x, y, y_1, y_2 \in L$ *and* $\pi(y_1)$ *the number of* h_i *equal to* y_1. *We then define:*

$C^3_*(L, M) = \left\{(\alpha, \beta),\ \alpha \in C^3_{CE}(L, M),\ \beta \text{ has the } (**)\text{-property w.r.t } \alpha\right\}$.

Positive characteristic - the restricted cohomology

We are in the following situation:

$$0 \longrightarrow C_*^0(L,M) \xrightarrow{d_*^0} C_*^1(L,M) \qquad C_*^2(L,M) \qquad C_*^3(L,M)$$

avec $d_*^0 = d_{CE}^0$.

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avec $d_*^0 = d_{CE}^0$.

$\rightsquigarrow$ it remains to build d_*^1 and d_*^2.

Positive characteristic - the restricted cohomology

An element $\varphi \in C^1_*(L,M)$ induces a map

$$\mathrm{ind}^1(\varphi)(x) = \varphi\left(x^{[p]}\right) - x^{p-1}\varphi(x), \quad x \in L$$

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We thus can define:

$$d_*^1(\varphi) = \left(d_{CE}^1\varphi, \operatorname{ind}^1(\varphi)\right).$$

Positive characteristic - the restricted cohomology

An element $(\alpha, \beta) \in C^2_*(L, M)$ induces a map

$$\mathrm{ind}^2(\alpha, \beta)(x, y) = \alpha\left(x, y^{[p]}\right) - \sum_{i+j=p-1} (-1)^i y^i \alpha\left([x, \overbrace{y, \dots, y}^{j \text{ terms}}], y\right) + x\beta(y).$$

We thus can define:

$$d^2_*(\alpha, \beta) = \left(d^2_{CE}\alpha, \mathrm{ind}^2(\alpha, \beta)\right).$$

Positive characteristic - the restricted cohomology

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$$0 \longrightarrow C^0_*(L,M) \xrightarrow{d^0_*} C^1_*(L,M) \xrightarrow{d^1_*} C^2_*(L,M) \xrightarrow{d^2_*} C^3_*(L,M)$$

avec $d^0_* = d^0_{CE}$.

Positive characteristic - the restricted cohomology

- A **restricted 2-cocycle** is an element $(\alpha, \beta) \in C^2_*(L, M)$ such that
  1. α is an ordinary Chevalley-Eilenberg 2-cocycle;
  2. $\alpha\left(x, y^{[p]}\right) - \sum_{i+j=p-1} (-1)^i y^i \alpha\left([x, \underbrace{y, \ldots, y}_{j \text{ terms}}], y\right) + x\beta(y) = 0.$

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- A **restricted 2-coboundary** is an element $(\alpha, \beta) \in C^2_*(L, M)$ such that $\exists \varphi \in \mathrm{Hom}(L, M)$,
  1. $\alpha(x, y) = \varphi([x, y]) - x\varphi(y) + y\varphi(x)$;
  2. $\beta(x) = \varphi\left(x^{[p]}\right) - x^{p-1}\varphi(x)$.

Restricted deformations

Definition

A restricted formal deformation of L is given by two maps

$$[\cdot,\cdot]_t : (x,y) \longmapsto \sum_{i\geq 0} t^i \mu_i(x,y), \quad (\cdot)^{[p]_t} : x \longmapsto \sum_{j\geq 0} t^j \omega_j(x),$$

with $\mu_0(x,y) = [x,y]$*,* μ_i *skewsymmetric,* $\omega_0 = (\cdot)^{[p]}$*,* ω_j *satisfying* $\omega_j(\lambda x) = \lambda^p \omega(x)$*.*
Moreover, $[\cdot,\cdot]_t$ *et* $(\cdot)^{[p]_t}$ *must satisfy*

$$[x,[y,z]_t]_t + [y,[z,x]_t]_t + [z,[x,y]_t]_t = 0; \tag{1}$$

$$\left[x, y^{[p]_t}\right]_t = [[\dots[x, \overbrace{y]_t, y]_t, \dots, y}^{p \text{ terms}}]_t. \tag{2}$$

Restricted deformations

In that framework, we recover all the usual results involving the cohomology up to order 2, for example:

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Proposition

Let $\left([\cdot,\cdot]_t, (\cdot)^{[p]_t}\right)$ *be a restricted formal deformation of* $(L, [\cdot,\cdot], [p])$.

1. *The map* ω_1 *has the* $(*)$*-property w.r.t.* μ_1 *and* (μ_1, ω_1) *is a 2-cocyle of the restricted cohomology.*

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Proposition

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1. *The map* ω_1 *has the* $(*)$*-property w.r.t.* μ_1 *and* (μ_1, ω_1) *is a 2-cocyle of the restricted cohomology.*
2. *The second cohomology group* $H^2_*(L, L)$ *classifies up to equivalence the infinitesimal restricted deformations of* L.

Lie-Rinehart algebras

In this slide, $\mathbb{K}$ is a characteristic 0 field.

Definition

A **Lie-Rinehart algebra** *is a pair* (A, L), *where*

- L *is a Lie* $\mathbb{K}$*-algebra, endowed with a bracket* $[\cdot, \cdot]$;
- A *is a commutative associative* $\mathbb{K}$*-algebra,*

such that, for $x, y \in L$ *and* $a, b \in A$:

- *There is an action* $A \times L \longrightarrow L,\ (a, x) \longmapsto a \cdot x$, *making* L *an* A*-module;*
- *There is a map* $\rho : L \longrightarrow Der(A),\ x \longmapsto \rho_x$, *which is both a Lie algebra morphism and a* A*-module morphism, called* **anchor map***;*
- $[x, a \cdot y] = \rho_x(a) \cdot y + a \cdot [x, y]$.

Restricted Lie-Rinehart Algebras.

Now $\mathbb{F}$ is again a characteristic p field. Let A be an associative commutative $\mathbb{F}$-algebra.

- $(A, \mathrm{Der}(A), \mathrm{id})$ is an Lie-Rinehart algebra;

Restricted Lie-Rinehart Algebras.

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- $(A, \mathrm{Der}(A), \mathrm{id})$ is an Lie-Rinehart algebra;
- $(\mathrm{Der}(A), (\cdot)^p)$ is a restricted Lie algebra;
- If $D \in \mathrm{Der}(A)$ and $a \in A$, we have

$$(aD)^p = a^p D^p + (aD)^{p-1}(a)D.$$

Restricted Lie-Rinehart Algebras.

Definition

Let A be an associative commutative algebra over a field $\mathbb{F}$ of characteristic p. Then (A, L) is a **restricted Lie-Rinehart algebra** *if*

1. *(A, L) is a Lie-Rinehart algebra, with anchor map $\rho : L \longrightarrow \mathrm{Der}(A)$;*
2. *$\left(L, (\cdot)^{[p]}\right)$ is a restricted Lie algebra;*
3. *$\rho(x^{[p]}) = \rho(x)^p$;*
4. *$(ax)^{[p]} = a^p x^{[p]} + \rho(ax)^{p-1}(a)x, \ a \in A, \ x \in L.$*

Thank you for your attention!