Symplectic double extensions for restricted quasi-Frobenius Lie (super)algebras

Quentin Ehret

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Joint work with Sofiane Bouarroudj and Yoshiaki Maeda

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Introduction

Definition

Let L be a $\mathbb{K}$ vector space. A **Lie bracket** *on L is a bilinear map $[\cdot,\cdot] : L \times L \longrightarrow L$ satisfying, for $x, y, z \in L$,*

1. $[x, y] = -[y, x]$ *(anticommutativity)*
2. $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ *(Jacobi identity).*

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- Double extensions of Lie algebras were introduced by Medina and Revoy (1985) in order to classify nilpotent Lie groups by their algebras.
- If L is a Lie algebra and $K = \text{span}\{X\}$, $K^* = \text{span}\{X^*\}$, a double extension of L is a Lie structure on $K \oplus L \oplus K^*$.

Introduction

- Benayadi, Bouarrouj, Hajli : Double extensions of restricted Lie superalgebra equipped with a non-degenerate invariant and symmetric bilinear form (2020),
  Double extensions of restricted Lie (super)algebras, Arnold. Math. J. **6** (2020), 231 – 269.

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- **Our goal :** "symplectic analog" of the first paper, that means, study double extensions of restricted quasi-Frobenius Lie superalgebras.
$\rightsquigarrow$ The cohomology involved is the *restricted cohomology*.

1. Introduction

2. Preliminaries
   - Restricted Lie superalgebras
   - Quasi-Frobenius Lie superalgebras
   - Derivations
   - Restricted cohomology

3. Symplectic Double Extensions
   - First case: orthosymplectic, even derivation
   - Converse

4. Examples
   - Example 1 : the Lie superalgebra $D^7_{q,-q}$
   - Example 2 : $K^{2,m}$, m odd

Lie superalgebras

Let $\mathbb{F}$ be a field of characteristic $p > 2$.

Definition

A Lie superalgebra $\mathfrak{a} = \mathfrak{a}_0 \oplus \mathfrak{a}_1$ *is a* $\mathbb{Z}/2\mathbb{Z}$*-graded vector space equipped with a bilinear map* $[\cdot, \cdot] : \mathfrak{a} \times \mathfrak{a} \to \mathfrak{a}$ *satisfying for* $a, b, c \in \mathfrak{a}$ *:*

1. $|[a, b]| = |a| + |b|$;
2. $[a, b] = -(-1)^{|a||b|}[b, a]$;
3. $(-1)^{|a||c|}[a, [b, c]] + (-1)^{|a||b|}[b, [c, a]] + (-1)^{|b||c|}[c, [a, b]] = 0.$

If $p = 3$, the identity $[a, [a, a]] = 0$, $a \in \mathfrak{a}_1$ has to be added as an axiom as well.

Restricted Lie algebras

Definition (Jacobson, 1941)

A **restricted Lie algebra** *is a Lie algebra* $\mathfrak{g}$ *equipped with a map* $(\cdot)^{[p]} : \mathfrak{g} \longrightarrow \mathfrak{g}$ *satisfying*

1. $(\lambda x)^{[p]} = \lambda^p x^{[p]}$, $x \in \mathfrak{g}$, $\lambda \in \mathbb{F}$;

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2. $\left[x, y^{[p]}\right] = [[\ldots[x, \overbrace{y], y], \ldots, y}^{p \text{ terms}}]$;
3. $(x + y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x, y)$,

with is$_i(x, y)$ *the coefficient of* Z^{i-1} *in* $\mathrm{ad}^{p-1}_{Zx+y}(x)$. *Such a map* $(-)^{[p]} : L \longrightarrow L$ *is called p-map.*

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Example : any associative algebra A with the commutator and $a^{[p]} := a^p$.
Example : restricted Heisenberg algebras.

Restricted Lie algebras

Very useful :

$$\sum_{i=1}^{p-1} s_i(x,y) = \sum_{\substack{x_i = x \text{ or } y \\ x_p = x,\ x_{p-1} = y}} \frac{1}{\sharp\{x\}} [x_1, [x_2, [\dots, [x_{p-1}, x_p]\dots],$$

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Definition

A Lie algebra morphism $f : (\mathfrak{g}, [\cdot,\cdot], (\cdot)^{[p]}) \to \left(\mathfrak{g}', [\cdot,\cdot]', (\cdot)^{[p]'}\right)$ *is said to be* **restricted** *if*

$$f\left(x^{[p]}\right) = f(x)^{[p]'},\; x \in \mathfrak{g}\,.$$

Restricted Lie superalgebras

Definition (Restricted Lie superalgebra)

A **restricted Lie superalgebra** *is a Lie superalgebra* $\mathfrak{a} = \mathfrak{a}_0 \oplus \mathfrak{a}_1$ *such that*

1. *The even part* $\mathfrak{a}_0$ *is a restricted Lie algebra;*
2. *The odd part* $\mathfrak{a}_1$ *is a Lie* $\mathfrak{a}_0$*-module;*
3. $\left[a, b^{[p]}\right] = [[\ldots[a, \overbrace{b], b], \ldots, b}^{p \text{ terms}}], \; a \in \mathfrak{a}_1, \; b \in \mathfrak{a}_0.$

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We can define a map $(\cdot)^{[2p]} : \mathfrak{a}_1 \to \mathfrak{a}_0$ *by*

$$a^{[2p]} = \left(a^2\right)^{[p]}, \; \text{with } a^2 = \frac{1}{2}[a, a], \; a \in \mathfrak{a}_1 .$$

One also says that $\mathfrak{a}$ has a $p|2p$ structure.

Quasi-Frobenius Lie superalgebras

Definition

A Lie superalgebra $\mathfrak{a}$ *is called* **quasi-Frobenius** *if it is equipped with a 2-cocycle* $\omega \in Z^2_{CE}(\mathfrak{a}; \mathbb{F})$ *such that* ω *is a non-degenerate bilinear form. Explicitly, for all* $a, b \in \mathfrak{a}$ *we have*

$$(-1)^{|a||c|}\omega(a, [b, c]) + (-1)^{|c||b|}\omega(c, [a, b]) + (-1)^{|b||a|}\omega(b, [c, a]) = 0.$$

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- If ω is even, $(\mathfrak{a}, \omega)$ is called **orthosymplectic.**

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- If ω is even, $(\mathfrak{a}, \omega)$ is called **orthosymplectic.**
- If ω is odd, $(\mathfrak{a}, \omega)$ is called **periplectic.**

(Restricted) derivations

Definition

Let $\mathfrak{a}$ be a restricted Lie superalgebra. A **derivation** *of $\mathfrak{a}$ is a linear map $D : \mathfrak{a} \to \mathfrak{a}$ such that*

$$D([a, b]) = [D(a), b] + (-1)^{|a||D|}[a, D(b)], \ a, b \in \mathfrak{a} .$$

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$$D(a^{[p]}) = (\mathrm{ad}_a)^{p-1}(D(a)),\ \ a \in \mathfrak{a}_0;$$

- A derivation D is said to have the p-**property** if there exists $\gamma \in \mathbb{F}$ and $a_0 \in \mathfrak{a}_0$ such that

$$D^p = \gamma D + \mathrm{ad}_{a_0},\ \text{ and }\ D(a_0) = 0.$$

(Restricted) derivations

If D is a derivation of a quasi-Frobenius Lie algebra $(\mathfrak{a}, \omega)$, there exists an unique linear map $D^* : \mathfrak{a} \to \mathfrak{a}$ satisfying the condition

$$\omega(D(a), b) = (-1)^{|a||(D|}\omega(a, D^*(b)), \ a, b \in \mathfrak{a}\,.$$

This map D^* is called the **adjoint** of D.

Moreover, D^* is a derivation as well.

Restricted cohomology

Let $\mathfrak{a}$ be a restricted Lie superalgebra and M a restricted $\mathfrak{a}$-module.

Definition

Let $\varphi \in C^2_{CE}(\mathfrak{a}, M)$ *et* $\theta : \mathfrak{a}_0 \longrightarrow M$. *We say that* θ **has the** $(*)$**-property w.r.t.** φ *if*

1. $\theta(\lambda a) = \lambda^p \theta(a),\ \lambda \in \mathbb{F},\ a \in \mathfrak{a}$;
2. $\theta(a+b) = \theta(a) + \theta(b) +$
$$\sum_{\substack{x_i \in \{a,b\} \\ x_1=a,\ x_2=b}} \frac{1}{\pi(a)} \sum_{k=0}^{p-2} (-1)^k x_p \dots x_{p-k+1} \varphi([[\dots[x_1, x_2], x_3] \dots, x_{p-k-1}], x_{p-k}),$$

with $a, b \in \mathfrak{a}$, $\pi(a)$ *the number of* x_i *equal to* a. *We then define*

$$C^2_*(\mathfrak{a}, M) = \left\{ (\varphi, \theta),\ \varphi \in C^2_{CE}(\mathfrak{a}, M),\ \theta \text{ has the } (*)\text{-property w.r.t. } \varphi \right\}.$$

Restricted cohomology

- A **restricted 2-cocycle** is an element $(\alpha, \beta) \in C^2_*(\mathfrak{a}, M)$ such that
  1. α is an ordinary Chevalley-Eilenberg 2-cocycle;
  2. $\alpha\left(a, b^{[p]}\right) - \sum_{i+j=p-1} (-1)^i y^i \alpha\left([a, \underbrace{b, \ldots, b}_{j \text{ terms}}], b\right) + a\beta(b) = 0, \ a, b \in \mathfrak{a}_0$.

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- A **restricted 2-coboundary** is an element $(\alpha, \beta) \in C^2_*(\mathfrak{a}, M)$ such that $\exists \varphi \in \mathrm{Hom}(\mathfrak{a}, M)$,
  1. $\alpha(a, b) = \varphi([a, b]) - a\varphi(b) + b\varphi(a),\ \ a, b \in \mathfrak{a}$;
  2. $\beta(a) = \varphi\left(a^{[p]}\right) - a^{p-1}\varphi(a),\ \ a \in \mathfrak{a}_0$.

Useful cocycles

Lemma

Let $(\mathfrak{a}, \omega)$ be a quasi-Frobenius Lie superalgebra and let D be a derivation. Let us define the map

$$C : \mathfrak{a} \wedge \mathfrak{a} \to \mathbb{F},$$

$$(a, b) \mapsto \omega\Big((D + D^*)(a), b\Big) = \omega\Big(D(a), b\Big) + \omega\Big(a, D(b)\Big).$$

Then, $C \in Z^2_{\mathrm{CE}}(\mathfrak{a}; \mathbb{F})$. Moreover, if D is inner, then $C \in B^2_{\mathrm{CE}}(\mathfrak{a}; \mathbb{F})$.

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Then, $C \in Z^2_{\mathrm{CE}}(\mathfrak{a}; \mathbb{F})$. Moreover, if D is inner, then $C \in B^2_{\mathrm{CE}}(\mathfrak{a}; \mathbb{F})$.

$\rightsquigarrow$ Now, we aim to build a map $P : \mathfrak{a}_0 \to \mathbb{F}$ such that (C, P) is a restricted cocycle.

Denote by $\sigma_i^{\mathfrak{a}}(a,b)$ the expression that appears in the following equation:

$$\omega\Big((D+D^*)(\mu a+b),(\mathrm{ad}^{\mathfrak{a}}_{\mu a+b})^{p-2}(a)\Big)=\sum_{1\leq i\leq p-1} i\sigma_i^{\mathfrak{a}}(a,b)\mu^{i-1}. \quad (1)$$

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Lemma

Let $P:\mathfrak{a}_o\to\mathbb{F}$ *be a map satisfying*

$$P(\delta a) = \delta^p P(a) \quad \text{for all } a\in\mathfrak{a}_0 \text{ and } \delta\in\mathbb{F}, \quad (2)$$

$$P(a+b) = P(a)+P(b)+\sum_{i=1}^{p-1}\sigma_i^{\mathfrak{a}}(a,b) \quad \text{for all } a,b\in\mathfrak{a}_0. \quad (3)$$

Then, P *has the* $(*)$*-property wrt the cochain* C *and the pair* (C,P) *is a restricted 2-cocycle if and only if*

$$\omega_{\mathfrak{a}}\Big((D+D^*)\left(b^{[p]}\right),a\Big)=\omega_{\mathfrak{a}}\Big((D+D^*)(b),\mathrm{ad}_b^{p-1}(a)\Big) \text{ for all } a,b\in\mathfrak{a}_0.$$

Symplectic Double Extensions

Let $(\mathfrak{a}, \omega)$ be a quasi-Frobenius Lie superalgebra.
We construct restricted double extensions $K \oplus \mathfrak{a} \oplus K^*$ of $\mathfrak{a}$, with $K = \text{span}\{X\}$ and $K^* = \text{span}\{X^*\}$. There are four cases to consider :

ω even (orthosymplectic case) → D even, D odd

ω odd (periplectic case) → D even, D odd

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- Let $(\mathfrak{a}, \omega)$ be an orthosymplectic Lie superalgebra and let D be an even restricted derivation satisfying the p-property. retour

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- For $\lambda \in \mathbb{F}$, consider the maps

$$\Omega : \mathfrak{a} \wedge \mathfrak{a} \to \mathbb{F}$$

$$(a, b) \mapsto \omega_{\mathfrak{a}}\Big(D \circ D(a) + 2D^* \circ D(a) + D^* \circ D^*(a) + \lambda(D + D^*)(a), b\Big)$$

$$T : \mathfrak{a}_0 \longrightarrow \mathbb{F}$$

$$a \longmapsto \omega_{\mathfrak{a}}\Big((D + D^*)(a), (\mathrm{ad}_a^{\mathfrak{a}})^{p-2} \circ D(a)\Big) + \lambda P(a).$$

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- Suppose that $(C, P) \in Z_*^2(\mathfrak{a}, \mathbb{F})$ and $(\Omega, T) \in B_*^2(\mathfrak{a}, \mathbb{F})$. Since ω is non-degenerate, there exists $Z \in \mathfrak{a}$ such that

$$\Omega(a, b) = \omega_{\mathfrak{a}}(Z, [a, b]_{\mathfrak{a}}), \ \forall a, b \in \mathfrak{a} \quad \text{and} \quad T(a) = \omega_{\mathfrak{a}}(Z, a^{[p]}), \ \forall a \in \mathfrak{a}_0 .$$

First case: orthosymplectic, even derivation

Theorem (Bouarroudj, E., Maeda (Part 1))

- *There exists a Lie superalgebra structure on* $\mathfrak{g} := K \oplus \mathfrak{a} \oplus K^*$, *defined as follows (for any* $a, b \in \mathfrak{a}$*):*

$$
\begin{aligned}
[x, x^*]_{\mathfrak{g}} &= \lambda x, \\
[a, b]_{\mathfrak{g}} &= [a, b]_{\mathfrak{a}} + C(a, b)x, \\
[x^*, a]_{\mathfrak{g}} &= D(a) + \omega_{\mathfrak{a}}(Z, a)x;
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$$[x^*, a]_{\mathfrak{g}} = D(a) + \omega_{\mathfrak{a}}(Z, a)x;$$

- *There exists a closed anti-symmetric orthosymplectic form* $\omega_{\mathfrak{g}}$ *on* $\mathfrak{g}$ *defined as follows (we only write non-zero-terms):*

$$\omega_{\mathfrak{g}}|_{\mathfrak{a} \times \mathfrak{a}} := \omega_{\mathfrak{a}}, \quad \omega_{\mathfrak{g}}(x^*, x) := 1.$$

First case: orthosymplectic, even derivation

Theorem (Bouarroudj, E., Maeda (Part 2))

- *There exists a $p|2p$-map on the double extension $\mathfrak{g}$ of $\mathfrak{a}$ given by*

$$\begin{array}{rcl} a^{[p]_{\mathfrak{g}}} & = & a^{[p]_{\mathfrak{a}}} + P(a)x, \\ (x^*)^{[p]_{\mathfrak{g}}} & = & \gamma x^* + a_0 + \tilde{\lambda} x, \\ x^{[p]_{\mathfrak{g}}} & = & b_0 + \sigma x + \delta x^*, \ \ \textit{where :} \end{array}$$

- *The case $\lambda \neq 0$:*

$$D(a_0) = 0, \quad \tilde{\lambda} = \frac{1}{\lambda}\omega(Z, a_0), \quad \gamma = \lambda^{p-1}, \qquad \delta = 0$$

$$D(b_0) = 0, \quad \sigma = \frac{1}{\lambda}\omega(Z, b_0), \quad D^*(b_0) = 0, \quad b_0 \textit{ central in } \mathfrak{a},$$

$$\textit{and } D^*(a_0) = \sum_{1 \leq i \leq p-1} (-1)^{p-1-i} \lambda^{p-1-i} {D^*}^i(Z_\Omega).$$

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Theorem (Bouarroudj, E., Maeda (Part 2))

- *There exists a* $p|2p$*-map on the double extension* $\mathfrak{g}$ *of* $\mathfrak{a}$ *given by*

$$\begin{aligned} a^{[p]_{\mathfrak{g}}} &= a^{[p]_{\mathfrak{a}}} + P(a)x, \\ (x^*)^{[p]_{\mathfrak{g}}} &= \gamma x^* + a_0 + \tilde{\lambda} x, \\ x^{[p]_{\mathfrak{g}}} &= b_0 + \sigma x + \delta x^*, \ \textit{where :} \end{aligned}$$

- *The case* $\lambda = 0$ *and* $D \neq -\delta^{-1}\,\mathrm{ad}_{b_0}$*:*

$$\begin{aligned} D(a_0) &= 0, & \omega(Z, a_0) &= 0, & \delta &= 0, \\ D(b_0) &= 0, & \omega(Z, b_0) &= 0, & D^*(b_0) &= 0, \quad b_0 \ \textit{central}, \end{aligned}$$

and

$$D^*(a_0) + \gamma Z = {D^*}^{p-1}(Z).$$

First case: orthosymplectic, even derivation

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- *The case $\lambda = 0$ and $D = -\delta^{-1} \mathrm{ad}_{b_0}$ is inner:*

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and

$$D^*(a_0) + \gamma Z_\Omega = (D^*)^{p-1}(Z).$$

First case: orthosymplectic, even derivation

Theorem (Bouarroudj, E., Maeda)

Let $(\mathfrak{g}, \omega_{\mathfrak{g}})$ be a restricted orthosymplectic quasi-Frobenius Lie superalgebra. Suppose there exists an even non-zero $x \in ([\mathfrak{g}, \mathfrak{g}]_{\mathfrak{g}})^{\perp}$ such that $K := Span\{x\}$ is an ideal, and $K^{\perp}$ is a p-ideal.

Then, $(\mathfrak{g}, \omega_{\mathfrak{g}})$ is obtained as a symplectic extension using an even derivation D from a restricted orthosymplectic quasi-Frobenius Lie superalgebra $(\mathfrak{a}, \omega_{\mathfrak{a}})$. Moreover, if the center of $\mathfrak{g}$ is non trivial, then we can choose x to be central.

(Quick) sketch of the proof

The "ordinary" part has been proven by Bouarroudj & Maeda (2021). We have $\mathfrak{g} = K^{\perp} \oplus K^{*}$. Define $\mathfrak{a} := (K \oplus K^{*})^{\perp}$, then we have $\mathfrak{g} = K \oplus \mathfrak{a} \oplus K^{*}$. In particular, we obtain that

$$\Omega \in B^2_{CE}(\mathfrak{a}, \mathbb{F}) \text{ and } C \in Z^2_{CE}(\mathfrak{a}, \mathbb{F}).$$

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Let's investigate the $p|2p$-mappings:

For $a \in \mathfrak{g}$, we have $a^{[p]_\mathfrak{g}} \in K^{\perp} = K \oplus \mathfrak{a}$. Therefore, it exists $s, P : \mathfrak{a} \to \mathbb{F}$ such that

$$a^{[p]_\mathfrak{g}} = s(a) + P(a)x.$$

$$\nearrow \qquad\qquad \nwarrow$$

$$p|2p - \text{map on } \mathfrak{a}\ ; \qquad\qquad (C, P) \in Z^2_*(\mathfrak{a}, \mathbb{F}).$$

Example 1 : the Lie superalgebra $D^7_{q,-q}$

Consider the $(2|2)$-dimensional restricted Lie superalgebra $D^7_{q,-q}$ $(q \neq 0, 1)$ given on the basis $(e_1, e_2 \mid e_3, e_4)$ by the brackets

$$[e_1, e_2] = e_2, \quad [e_1, e_3] = qe_3, \quad [e_1, e_4] = -qe_4,$$

and the $p|2p$-map $e_1^{[p]} = e_1$ and $e_2^{[p]} = 0$.

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This superalgebra is orthosymplectic quasi-Frobenius with the form

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$$\begin{cases} [x, x^*]_{\mathfrak{g}} & = -x, \\ [a, b]_{\mathfrak{g}} & = [a, b]_{\mathfrak{a}} + C(a, b)x, \\ [x^*, a]_{\mathfrak{g}} & = D_1(a) + \omega_{\mathfrak{a}}(ue_2, a)x; \end{cases}$$

$$\begin{cases} e_1^{[p]_{\mathfrak{g}}} & = e_1^{[p]_{\mathfrak{a}}} + ux, \\ e_2^{[p]_{\mathfrak{g}}} & = 0, \\ (x^*)^{[p]_{\mathfrak{g}}} & = x^*, \\ x^{[p]_{\mathfrak{g}}} & = 0. \end{cases}$$

Example 2 : the Lie superalgebra $K^{2,m}$, m odd

The Lie superalgebra $K^{2,m}$ (Gomez, Khakimdjanov, Navarro) is spanned by the generators $(x_0, x_1 \mid y_1, \dots y_m)$ (Even | Odd), with non-zero brackets given by

$$\begin{aligned} [x_0, y_i] &= -[y_i, x_0] &= y_{i+1}, \quad & i \leq m-1, \\ [y_i, y_{m+1-i}] &= [y_{m+1-i}, y_i] &= (-1)^{i+1} x_1, \quad & 1 \leq i \leq \tfrac{m+1}{2}. \end{aligned}$$

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Proposition

- *$K^{2,m}$ is orthosymplectic quasi-Frobenius if and only if $m = 0$ mod (p). In that case, the form is given by*

$$x_0^* \wedge x_1^* - \frac{1}{2}\, y_1^* \wedge y_1^* - \frac{1}{2}\,(-1)^{\frac{m+3}{2}}\, y^*_{\frac{m+1}{2}} \wedge y^*_{\frac{m+3}{2}} - \sum_{1 \leq i \leq \frac{m-3}{2}} i\,(-1)^{i+1}\, y^*_{i+1} \wedge y^*_{m+1-i}.$$

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- *$K^{2,m}$ is restricted if and only if $m \leq p$, with the $[p|2p]$-map given by $x_0^{[p]} = s_1 x_1, \quad x_1^{[p]} = s_2 x_1$, where $s_1, s_2 \in \mathbb{F}$.*

$\rightsquigarrow$ Hereafter, we will consider $x_0^{[p]} = 0, \quad x_1^{[p]} = x_1$.

Example 2 : $K^{2,m}$, m odd

A derivation yielding a trivial extension.

Consider the outer restricted derivation given by

$$D = x_1 \otimes x_0^*.$$

We have $D^* = -D$. It follows that that the cocycle C as well as the map Ω are identically zero.

Therefore, the double extension is trivial.

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A derivation yielding a non-trivial extension.

Consider the outer restricted derivation given by

$$D = y_{p-1} \otimes y_1^* + y_p \otimes y_2^*; \qquad D^* = y_p \otimes y_2^* - 2y_1 \otimes y_3^*.$$

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It follows that

$$\Omega(y_1, y_3) = -2\lambda, \quad \Omega(y_2, y_2) = 2\lambda.$$

The case $p = 3$: $\Omega = d^2_{CE}(\lambda\, x_1^*)$ and $C = \frac{1}{2}\, y_1^* \wedge y_3^* + y_2^* \wedge y_2^*$;

$$Z = \lambda x_0, \quad \text{and} \quad P(a) = x_1^*(a^{[p]}) \text{ for } a \in K_0^{2,3},$$

$$\gamma = \lambda^{p-1}, \quad a_0 = -\gamma x_0, \quad \tilde{\lambda} = 0, \quad b_0 = x_1, \quad \sigma = 1.$$

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It follows that

$$\Omega(y_1, y_3) = -2\lambda, \quad \Omega(y_2, y_2) = 2\lambda.$$

The case $p > 3$: The map Ω cannot be a coboundary, except for $\lambda = 0$ where it becomes identically trivial. In this case, we can choose

$$\gamma = 0, \quad b_0 = 0, \quad a_0 = x_1, \quad Z = x_1.$$

Thank you for your attention!