Midterm Exam

1. Decompose $1+3i$ in $\mathbb{Z}[i]$.

2. Let $\alpha = \frac{1}{2}(2+\sqrt{5})$. Check whether α is an algebraic number or not.

3. Let R be a ring and let J and K be two ideals of R such that $J \subset K$.
   (a) Show that K/J is an ideal of R/J.
   (b) Show the *third isomorphism theorem :*
   $$\frac{R/J}{K/J} \cong R/K.$$
   (c) Use question 3(b) to show that $\mathbb{Z}[x]/(x-1,3) \cong \mathbb{F}_3$.

4. In the ring $\mathbb{Z}/6\mathbb{Z}$, show that $a=2$ is prime and reducible. Explain why in this ring, a prime element can be reducible.

5. Let $P(x) = 12x^3 + 18x^2 + 6x \in \mathbb{Z}[x]$. Decompose P into prime elements in $\mathbb{Z}[x]$.

6. Let A be the ideal of $\mathbb{Z}[\sqrt{-5}]$ generated by $(3, 1+\delta)$, with $\delta = \sqrt{-5}$. Show that $A^2 = (2-\delta)$.

7. Let L be the lattice in the plane generated by the vectors $u=(1,2)^t$ and $v=(2,1)^t$. Let $\Delta(L)$ be the area of the parallelogram spanned by u and v. Find a vector $w \in L$ such that $|w|^2 \leq \frac{2}{\sqrt{3}}\,\Delta(L)$.