Problems 1

1 Exercises from Artin

p. 354 : ex 1.3 ;
p. 355 : ex 3.8, 3.9, 3.11, 4.3, 4.4.

2 Rings and homomorphisms

1. Let R be a ring. The characteristic if R is the generator of the kernel of the map $\mathbb{Z} \to R$ (which is a principal ideal). Determine all rings of cardinality p and characteristic p.
2. Let R be a commutative ring. Let $Nil(R) = \{r \in R,\ \exists n \geq 1,\ r^n = 0\}$.
   (a) Prove that $Nil(R)$ is an ideal of R.
   (b) Let $r \in Nil(R)$, show that $1 - r$ is invertible in R.
   (c) In the case where R is not commutative, show that $Nil(R)$ is not necessarely an ideal.

3 Quotient rings

1. Show that $\mathbb{Z}[i]/(1+i) \cong \mathbb{Z}/2\mathbb{Z}$.
2. Show that $\mathbb{Z}[x]/(n,x) \cong \mathbb{Z}/n\mathbb{Z},\ n \geq 2$.
3. Show that $\mathbb{Z}[x]/(n) \cong \mathbb{Z}/n\mathbb{Z}[x],\ n \geq 2$.

4 Polynomials

1. Let $\mathbb{K}$ be a field. Show that a polynomial of degree 2 or 3 in $\mathbb{K}[x]$ is irreducible if and only if it doesn't admit any root in $\mathbb{K}$.
2. Find all irreducible polynomials of degree 2 and 3 with $\mathbb{K} = \mathbb{Z}/2\mathbb{Z}$.
3. Show that the polynomials $5x^3 + 8x^2 + 3x + 15$ and $x^5 + 2x^3 - 6x - 5$ are irreducible in $\mathbb{Z}$.
4. Describe all polynomials of degree 4 and 5 on $\mathbb{Z}/2\mathbb{Z}$.
5. Let $P \in \mathbb{Z}[x]$. Suppose that $P(0)$ and $P(1)$ are odd. Show that P doesn't have any root in $\mathbb{Z}$.
6. More generally, suppose that n doesn't divide any of the numbers $P(0), P(1), \cdots P(n-1)$. how that P doesn't have any root in $\mathbb{Z}$.