Problems 5

1. Let $\delta = \sqrt{-5}$, $R = \mathbb{Q}[\delta]$ and let $B = (3, 1+\delta)$. Find a generator for the principal ideal B^2.

2. Let $R = \mathbb{Q}[\delta]$, with $\delta = \sqrt{-21}$. We denote by G the class group.
   (a) Explain why we only have to consider the primes $2, 3, 5$ to compute G.
   (b) Show that 2 ramifies. Deduce a first relation within G.
   (c) Show that 3 and 5 split.
   (d) By computing the norm of some well-chosen elements, find two extra relations within G.
   (e) Conclude that G is isomorphic to the Klein group $C_2 \times C_2$.

   The Klein group can be interpreted as the group of symmetries of a "regular" cross (see below). It is the smallest non-cyclic group.

3. Let $\delta = \sqrt{-3}$ and $R = \mathbb{Z}[\delta]$. Note that since $-3 = 1[4]$, R is not the ring of integers of $\mathbb{Q}[\delta]$. Let $A = (2, 1+\delta)$.
   (a) Prove that R/A is a field and that A is maximal.
   (b) Prove that $\overline{A}A$ is not principal and that the Main Lemma is not valid.
   (c) Prove that A contains the ideal (2), but that A doesn't divide (2). Compare with Corollary 13.4.9 of Artin. Comments ?