Problems 6

1. Let $F \subset K \subset L$ be fields. Show that $[L : F] = [L : K][K : F]$.

2. Let $F \subset K$ be a field extension of degree n. Let $\alpha \in K$. Show that α is algebraic over F and that its degree divides n.

3. Let $F \subset F' \subset L$ be fields. Suppose that $\alpha \in L$ is algebraic over F of degree d. Show that α is algebraic over F' of degree $\leq d$.

4. Show that a field extension K/F generated by finitely many algebraic elements is a finite extension.

5. Let K/F be an extension of prime degree p and let $\alpha \in K,\ \alpha \notin F$. Show that α has degree p and that $F(\alpha) = K$.

6. Let L/F be a field extension, let K and F be subfields of L that are finite extensions of F, and let K' be the subfield of L generated by K and F' together. Let $[K' : F] = N,\ [K : F] = m,\ [F' : F] = n$. Draw a diagram summing up the situation, show that n and m divide N and that $N \leq mn$.

7. (a) Let $\alpha_1, \alpha_2, \alpha_3$ denote the complex roots of $x^3 - 2$. Show that $[\mathbb{Q}(\alpha_1, \alpha_2) : \mathbb{Q}] = 6$.
   (b) Let $K = \mathbb{Q}(\sqrt{2}, i)$. Show that $[\mathbb{Q}(\sqrt{2}, i) : \mathbb{Q}] = 4$.