Problems 2

1. Let $H \subset G$ be a subgroup of G. Show that the following assertions are equivalent :
   - (i) H is a normal subgroup of G ;
   - (ii) $H = gHg^{-1},\ \forall g \in G$;
   - (iii) $ghg^{-1} \in H,\ \forall g \in G,\ \forall h \in H$.

2. Let $\varphi : G \to H$ be a group homomorphism.
   - (a) Show that $\mathrm{Im}(\varphi)$ is a subgroup of H.
   - (b) Show that $\ker(\varphi)$ is a normal subgroup of G.
   - (c) Show that $\ker(\varphi) = 1_G \iff \varphi$ is injective.

3. Let $H \subset G$ such that $[G : H] = 2$. Show that H is a normal subgroup of G.

4. Let $n \geq 3$. Show that $Z(S_n) = \{\mathrm{id}\}$.