Problems 5

1. Consider the map

$$\begin{aligned} f : \mathbb{R}_n[X] &\to \mathbb{R}_n[X] \\ P &\mapsto (X^2-1)P'(X) - (nX+1)P(X). \end{aligned}$$

   (a) Show that the map f is well-defined.
   (b) For $0 \leq k \leq n$, consider $P_k(X) = (1-X)^k(1+X)^{n-k}$. Compute P_k', then $f(P_k)$.
   (c) Deduce that f is diagonalizable and bijective.

2. Let G be a group. For any $g \in G$, consider the map $\varphi_g : x \mapsto gxg^{-1}$. We denote by

$$\mathrm{Int}(G) := \{\varphi_g,\ g \in G\}.$$

   (a) Show that $\mathrm{Int}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$.
   (b) Consider the map $f : G \to \mathrm{Int}(G), \quad g \mapsto \varphi_g$. Show that f is a group homomorphism and compute its kernel.
   (c) Show that $G/Z(G) \cong \mathrm{Int}(G)$.

3. Let G, H be groups of finite order such that $|G| = m,\ |H| = n$ with $\gcd(m,n) = 1$. Determine all the group homomorphisms $f : G \to H$.

4. Find all group homomorphisms $(\mathbb{Q}, +) \to (\mathbb{Z}, +)$.

5. Let G be a group. An element $x \in G$ is called torsion element if $\exists\ n > 0,\ x^n = e$. The group G is called *torsion group* if all its elements are torsion elements and *torsion free* if the only torsion element of G is e.
   Let G_1 be a torsion group and G_2 a torsion free group. Find all group isomorphisms $G_1 \to G_2$.