Final Exam

We are working over fields of characteristic 0.

Problem 1

1. (a) Let $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Show that the extension $K/\mathbb{Q}$ is Galois and describe its Galois group.
   (b) Let $\varepsilon = -3 + \frac{3\sqrt{2}}{2} + \sqrt{3} - \frac{\sqrt{6}}{2} \in K$. Consider the maps $\sigma_1, \sigma_2 : K \to K$ defined by

$$\sigma_1(\sqrt{3}) = -\sqrt{3},\ \sigma_1(\sqrt{2}) = \sqrt{2};\quad \sigma_2(\sqrt{2}) = -\sqrt{2},\ \sigma_2(\sqrt{3}) = \sqrt{3};\quad \sigma_1|_{\mathbb{Q}} = \sigma_2|_{\mathbb{Q}} = \mathrm{id}\,.$$

   Show that :
   (i) $\sigma_1(\varepsilon) = \varepsilon\, a^2$, where $a = \frac{\sqrt{2}}{2} + \frac{\sqrt{6}}{2}$;
   (ii) $a\sigma_1(a) = -1$;
   (iii) $\sigma_2(\varepsilon) = \varepsilon\, b^2$, where $b = 1 + \sqrt{2}$;
   (iv) $b\sigma_2(b) = -1$.
2. Let K/F a Galois extension satisfying $[K : F] = 2$; and let $\mathrm{Gal}(K/F) = \{\mathrm{id}, \sigma\}$. Suppose that there exists an element $\varepsilon \in K$ satisfying $\sigma_1(\varepsilon) = \varepsilon\, a^2$ where $a \in K$ is such that $a\sigma_1(a) = -1$. Let $L = K(\sqrt{\varepsilon})$.
   (a) Show that L/F is Galois.
   (b) Explain why there exists $\tau \in \mathrm{Gal}(L/F)$ such that $\tau|_K = \sigma$. Explain why τ cannot be an element of order 2.
   (c) Deduce that $\mathrm{Gal}(L/F)$ is cyclic of order 4.
3. In this question, K and ε are defined as in Question 1. Let $F_1 = \mathbb{Q}(\sqrt{2})$, $F_2 = \mathbb{Q}(\sqrt{3})$ and $L = K(\sqrt{\varepsilon})$.
   (a) Draw a clear diagram of the situation.
   (b) For $i \in \{1, 2\}$, we denote H_i the group generated by σ_i (σ_1 and σ_2 are given in 1.(b)). Show that $F_i = K^{H_i}$, where K^{H_i} is the field of the elements fixed by H_i. Deduce a description of $\mathrm{Gal}(K/F_i)$.
   (c) Using Question 2, describe $\mathrm{Gal}(L/F_i)$, $i \in \{1, 2\}$.
   (d) Show that $L/\mathbb{Q}$ is Galois.
   (e) Describe $\mathrm{Gal}(L/\mathbb{Q})$.

Problem 2

Notation : $a^{1/n} \equiv \sqrt[n]{a}$.
A field L is *algebraically closed* if every polynomial equation with coefficients in L admits a root in L. For a finite Galois extension K/F, we define the 'norm'

$$N : K \longrightarrow F$$
$$x \longmapsto \prod_{\sigma \in \mathrm{Gal}(K/F)} \sigma(x).$$

1. Show that the map N is well defined, that is, $N(x) \in F \quad \forall x \in K$.
2. Show that $N(xy) = N(x)N(y)\ \forall x, y \in K$.
3. Let $a \in F^\times \setminus (F^\times)^p$, where p is prime. Suppose that $K = F(a^{1/p})$ and that F contains a primitive p-th root of the unity. Describe $\mathrm{Gal}(K/F)$ and show that $\sigma \in \mathrm{Gal}(K/F) \Rightarrow \sigma(a^{1/p}) = \omega a^{1/p}$, where $\omega^p = 1$. Deduce that $N(a^{1/p}) = \omega^{p(p-1)/2} a$. What happens the case where p is odd ?
4. In the same context, show that K is not algebraically closed in general if p is odd.