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Classification and deformations of Lie-Rinehart superalgebras

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Introduction

Lie-Rinehart algebras are algebraic analogs of Lie algebroids. They first appeared in the work of Rinehart and have been studied by Huebschmann. The notion of Lie-Rinehart superalgebra is a generalization of Lie-Rinehart algebras to the superworld. Such structures appear in the study of superalgebras of differentials operators on a supermanifold and can be used to generalize envelopping superalgebras related to those supermanifolds. The purpose of this work is to determine the structure laws of Lie-Rinehart superalgebras over $\mathbb{C}$ in low dimensions and to develop a theory of formal deformations of those structures.

Lie-Rinehart superalgebras

$\mathbb{K}$ is a characteristic zero field.

Definition

A **Lie-Rinehart superalgebra** is a pair (A, L), where

- $L = L_0 \oplus L_1$ is a Lie superalgebra over $\mathbb{K}$, endowed with a bracket $[\cdot,\cdot]$;
- $A = A_0 \oplus A_1$ is an associative and supercommutative $\mathbb{K}$-superalgebra,

such that, for $x, y \in L$ and $a \in A$:

1. There is an action $A \times L \longrightarrow L,\ (a, x) \longmapsto a \cdot x$, making L an A-module;
2. There is an action of L on A by superderivations: $L \longrightarrow \text{Der}(A),\ x \longmapsto (\rho_x : a \longmapsto \rho_x(a))$, such that ρ is an A-linear even morphism of Lie superalgebras (ρ is called **anchor map**);
3. Compatibility condition: $[x, ay] = \rho_x(a)y + (-1)^{|a||x|}a[x, y]$.

Example: Lie superalgebra of superderivations.
Let A be an associative unital superalgebra, and $L = \text{Der}(A)$ be its superalgebra of superderivations. The pair $(A, \text{Der}(A))$ is a Lie-Rinehart superalgebra, with the trivial anchor given by $\rho(\delta) = \delta,\ \forall \delta \in \text{Der}(A)$.

Classification

We aim to study $\mathbb{C}$-Lie-Rinehart superstructures in the case $\dim(A) \leq 2$ and $\dim(L) \leq 4$. For any pair (A, L), we compute the compatible actions and anchors. We denote basis elements of A_0 by e_i^0 and those of A_1 by e_j^1. The unit is e_1^0. For Lie superalgebras, we denote basis elements of L_0 by f_i^0 and those of L_1 by f_j^1. We say that an action is **trivial** if $e_i^s \cdot f_j^t = f_j^t$ if $i = 1, s = 0$, and 0 otherwise.

Lemma

If A is an associative superalgebra and L a Lie superalgebra, then we can always endow the pair (A, L) with a super Lie-Rinehart structure using the trivial action and the null anchor.

Example: If $\dim(A) = \dim(L) = (1|1)$, there are three pairwise non-isomorphic Lie superalgebras $\mathbf{L}^1_{1|1} : [\mathbf{f}_1^1, \mathbf{f}_1^1] = \mathbf{f}_1^0$; $\mathbf{L}^2_{1|1} : [\mathbf{f}_1^0, \mathbf{f}_1^1] = \mathbf{f}_1^1$ and $\mathbf{L}^3_{1|1}$: null bracket and only one associative superalgebra $\mathbf{A}^1_{1|1}$ with product $e_1^1 e_1^1 = 0$. We obtain the following table ($\lambda \in \mathbb{C}$):

A	L	Action	Anchor
$A^1_{1\vert 1}$	$L^1_{1\vert 1}$	$e_1^1 \cdot f_1^1 = \lambda f_1^0$	null
	$L^2_{1\vert 1}$	trivial	$\rho(f_1^0)(e_1^1) = \lambda e_1^1$
		$e_1^1 \cdot f_1^1 = \lambda f_1^0$	$\rho(f_1^0)(e_1^1) = -e_1^1,\ \rho(f_1^1)(e_1^1) = -\lambda e_1^0$
		$e_1^1 \cdot f_1^0 = \lambda f_1^1$	$\rho(f_1^0)(e_1^1) = e_1^1$
	$L^3_{1\vert 1}$	trivial	$\rho(f_1^0)(e_1^1) = \lambda e_1^1$
		$e_1^1 \cdot f_1^0 = \lambda f_1^0$	null
		$e_1^1 \cdot f_1^0 = \lambda f_1^1$	null

Proposition

Let (A, L) be a Lie-Rinehart superalgebra, with $\dim(A) \leq 2$ and L abelian. Then either the action is trivial or the anchor is null.

Formal deformations

Definition (Super-multiderivations space)

If M is a L-module, we define $\text{Der}^n(M, M)$ as the space of multilinear maps

$$f : M^{\otimes n+1} \longrightarrow M$$

such that it exists $\sigma_f : M^{\otimes n} \longrightarrow \text{Der}(A)$ (called symbol map) and

1. $f(x_1, ..., x_{n+1}) = -(-1)^{|x_i||x_{i+1}|} f(x_1, ..., x_{i+1}, x_i, ..., x_{n+1}),\ \forall 1 \leq i \leq n,$
2. $f(x_1, ...x_i, ...ax_{n+1}) = (-1)^{|a|(|f|+|x_1|+...+|x_n|)} a f(x_1, ..., x_{n+1}) + \sigma_f(x_1, ..., \hat{x}_i, ..., x_{n+1})(a)(x_i),\ a \in A.$

Remark: With this definition, we can check that if (A, L) is a Lie-Rinehart superalgebra, the bracket $[\cdot,\cdot]$ on L belongs to $\text{Der}^1(L, L)$, with symbol map given by the anchor.

Proposition

The space $\text{Der}^*(M, M) = \oplus_{n \geq -1} \text{Der}^n(M, M)$ carries a $\mathbb{Z}$-graded Lie algebra structure.

Proposition

There is a one-to-one correspondence between Lie-Rinehart superstructures on (A, L) and elements $m \in \text{Der}^1(L, L)$ such that $[m, m] = 0$.

We set $C^n_{def}(L, L) := \text{Der}^{n-1}(L, L)$ and endow it with a differential map

$$\delta : C^n_{def}(L, L) \longrightarrow C^{n+1}_{def}(L, L),\ D \longmapsto [m, D].$$

Thus, we can define the **deformation cohomology** with this complex.

Definition

Let $(A, L, [\cdot,\cdot], \rho)$ be a Lie-Rinehart superalgebra, and let $m \in \text{Der}^1(L, L)$ be the corresponding super-multiderivation. A deformation of a Lie-Rinehart superstructure is a $\mathbb{K}[[t]]$-bilinear map

$$m_t : L \times L \longrightarrow L[[t]],$$
$$(x, y) \longmapsto \sum_{i \geq 0} t^i m_i(x, y),$$

$m_0 = m$ and $m_i \in \text{Der}^1(L, L)$, with symbol map denoted by σ_{m_i}.
Moreover, m_t must verify $[m_t, m_t] = 0$, the bracket being the $\mathbb{Z}$-graded bracket on $\text{Der}^*(L[[t]], L[[t]])$.

Deformation equation. Since m_t satisfies $[m_t, m_t] = 0$, we have

$$\forall k \geq 0,\ \sum_{i \geq 0}^{k} m_i(x_1, m_{k-i}(x_2, x_3)) = \sum_{i \geq 0}^{k} m_i(m_{k-i}(x_1, x_2), x_3) + (-1)^{|x_1||x_2|} \sum_{i \geq 0}^{k} m_i(x_2, m_{k-i}(x_1, x_3)).$$

Theorem

Let m_t be a deformation of a Lie-Rinehart superalgebra (A, L). Then the infinitesimal of the deformation m_1 is a 2-cocycle of the deformation cohomology.

A Lie-Rinehart superalgebra is said to be **rigid** if every deformation is trivial, *i.e.* equivalent to the deformation $m_t = m$ along a formal automorphism

$$\varphi_t = \text{id} + \sum_{i \geq 1} \varphi_i.$$

Theorem

Any non-trivial deformation of $m \in Der^1(L, L)$ is equivalent to a deformation whose infinitesimal is not a coboundary.

Example: The Lie-Rinehart superalgebra $\left(\mathbf{A}^1_{1|1}, \mathbf{L}^1_{1|1}\right)$ is rigid.

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