Study and deformations of Lie-Rinehart algebras in positive characteristic

PhD Thesis Defense

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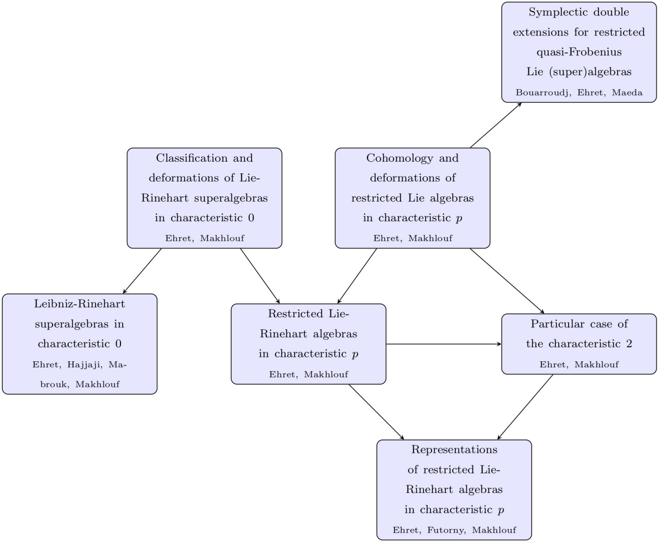
Département de Mathématiques

Overview of the work



Non-associative algebras and superalgebras

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(1842-1899)

- **associative :** $a(bc) = (ab)c, \ \forall a, b, c \in A$;
- **commutative :** $ab = ba, \ \forall a, b \in A$;
- **Lie algebra :**
  1. $ab = -ba, \ \forall a, b \in A$;
  2. $a(bc) + b(ca) + c(ab) = 0, \ \forall a, b, c \in A$ (Jacobi).

Notation for Lie algebras : $ab =: [a, b], \ a, b \in A$.

Non-associative algebras and superalgebras

Let $(A = A_0 \oplus A_1, +)$ be a super vector space over a field $\mathbb{K}$. It is called **(non-associative) superalgebra** if it is endowed with a multiplicative law $A_i \times A_j \longrightarrow A_{i+j \mod 2}$.

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- **associative :** $a(bc) = (ab)c, \ \forall a, b, c \in A;$
- **supercommutative :** $ab = (-1)^{|a||b|} ba, \ \forall a, b \in A;$
- **Lie superalgebra :**
  1. $ab = -(-1)^{|a||b|} ba, \ \forall a, b \in A;$
  2. $(-1)^{|a||c|} a(bc) + (-1)^{|a||b|} b(ca) + (-1)^{|c||b|} c(ab) = 0,$ $\forall a, b, c \in A$ (Jacobi).

Notation for Lie (super)algebras : $ab =: [a, b], \ a, b \in A.$

Brief history of Lie-Rinehart (super)algebras

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Definition

A **Lie-Rinehart superalgebra** *on a field* $\mathbb{K}$ *is a triple* (A, L, ρ), *with* $(L, [\cdot,\cdot])$ $\mathbb{K}$*-Lie superalgebra and* A *an associative supercommutative* $\mathbb{K}$*-superalgebra, such that* L *is an* A*-module, and there is a map*

$$\rho : L \longrightarrow Der(A), \quad x \mapsto \rho_x$$

called **anchor**, *which is an* A*-modules map and a Lie superalgebras map. Moreover, it has to satisfy the* **compatibility condition**

$$[x, a\cdot y] = \rho_x(a)\cdot y + (-1)^{|a||x|} a\cdot[x,y], \; \forall x,y \in L, \; \forall a \in A.$$

$$\mathrm{Der}(A) = \left\{ f : A \to A, \; f(ab) = f(a)b + (-1)^{|f||a|} af(b) \right\}.$$

Classification: an example

(1|1, 1|1)-type: $(\alpha_i \in \mathbb{C},\ \alpha_i \neq 0)$

A	L	Action	Anchor
$A^1_{1\|1}$	$L^1_{1\|1}$	$e^1_1 \cdot f^1_1 = \alpha_1 f^0_1$	null
	$L^2_{1\|1}$	trivial	$\rho(f^0_1)(e^1_1) = \alpha_2 e^1_1$
		$e^1_1 \cdot f^1_1 = \alpha_3 f^0_1$	$\rho(f^0_1)(e^1_1) = -e^1_1,\ \rho(f^1_1)(e^1_1) = -\alpha_3 e^0_1$
		$e^1_1 \cdot f^0_1 = \alpha_4 f^1_1$	$\rho(f^0_1)(e^1_1) = e^1_1$
	$L^3_{1\|1}$	trivial	$\rho(f^0_1)(e^1_1) = \alpha_5 e^1_1$
		$e^1_1 \cdot f^0_1 = \alpha_6 f^1_1$	null

$A^1_{1|1} = \langle e^0_1, e^1_1,\ e^1_1 e^1_1 = 0\rangle;$

$L^1_{1|1} = \langle f^0_1, f^1_1; [f^1_1, f^1_1] = f^0_1\rangle;$ $\quad L^2_{1|1} = \langle f^0_1, f^1_1; [f^0_1, f^1_1] = f^1_1\rangle;$

$L^3_{1|1} = \langle f^0_1, f^1_1;\ [\cdot,\cdot] = 0\rangle.$

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$\rightsquigarrow$ We have obtained all Lie-Rinehart superalgebras structures on pairs (A, L) with $\dim(A) \leq 2$ and $\dim(L) \leq 4$.

Deformation theory: Super-multiderivations

Let (A, L, ρ) be a Lie-Rinehart superalgebra and M an A-module.

Definition (Super-multiderivations space)

We define $\mathrm{Der}^n(M, M)$ *as the space of multilinear maps*

$$f : M^{\wedge(n+1)} \longrightarrow M$$

such that it exists an application $\sigma_f : M^{\times n} \longrightarrow \mathrm{Der}(A)$ *(called symbol map), such that*

$$\sigma_f(x_1, \cdots, a \cdot x_i, \cdots, x_n) = (-1)^{|a|(|x_1|+\cdots+|x_{i-1}|)} a \cdot \sigma_f(x_1, \cdots, x_i, \cdots, x_n);$$
$$f(x_1, \cdots, x_n, a \cdot x_{n+1}) = (-1)^{|a|(|f|+|x_1|+\cdots+|x_n|)} a \cdot f(x_1, \cdots, x_{n+1}) + \sigma_f(x_1, \cdots, x_n)(a)(x_{n+1}), \ \forall a \in A.$$

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Lie structure: $f \in \mathrm{Der}^p(M,M)$ and $g \in \mathrm{Der}^q(M,M)$:

$$[f,g] = f \circ_G g - (-1)^{pq} g \circ_G f,$$
$$\text{with symbol map } \sigma_{[f,g]} = \sigma_f \circ_G g - (-1)^{pq} \sigma_g \circ_G f + [\sigma_f, \sigma_g],$$

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with

$$(f \circ_G g)(x_1, \cdots, x_{p+q+1}) = \sum_{\tau \in Sh(q+1,p)} \varepsilon(\tau, x_1, \cdots, x_{p+q+1}) \times f\left(g(x_{\tau(1)}, \cdots, x_{\tau(q+1)}), x_{\tau(q+2)}, \cdots, x_{\tau(p+q+1)}\right).$$

Deformation theory: Super-multiderivations

Proposition

There is a one-to-one correspondence between Lie-Rinehart superalgebras structures on (A, L) *and elements* $m \in \mathrm{Der}^1(L, L)$ *such that* $[m, m] = 0$.

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$\rightsquigarrow$ we can construct a deformation theory by identifying a Lie-Rinehart supealgebra structure on (A, L) and the corresponding element $m \in \mathrm{Der}^1(L, L)$.

Deformation theory: Deformation cohomology

Cochains space.

$$C^n_{def}(L, L) := \mathrm{Der}^{n-1}(L, L)$$
$$C^*_{def}(L, L) := \bigoplus_{n \geq 0} C^n_{def}(L, L).$$

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$$\delta^n : C^n_{def}(L,L) \longrightarrow C^{n+1}_{def}(L,L),$$
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$(C^*_{def}(L, L), \delta^*)$ is a cochain complex.

We define $H^n_{def}(L, L) = \ker(\delta^n)/\operatorname{im}(\delta^{n-1})$.

Deformation theory: main results

Definition

Let (A, L, ρ) be a Lie-Rinehart superalgebra, and let $m \in \mathrm{Der}^1(L, L)$ be the corresponding super-multiderivation. A deformation of m is given by

$$\begin{aligned} m_t : L \times L &\longrightarrow L[[t]] \\ (x, y) &\longmapsto \sum_{i \geq 0} t^i m_i(x, y), \quad m_0 = m,\ m_i \in \mathrm{Der}^1(L, L), \end{aligned}$$

Moreover, m_t must verify $[m_t, m_t] = 0$, the bracket being the $\mathbb{Z}$-graded bracket on $C^*_{def}(L[[t]], L[[t]])$.

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Theorem

- *Let m_t be a deformation of a Lie-Rinehart superalgebra (A, L, ρ). Then the infinitesimal m_1 is a 2-cocycle with respect to the deformation cohomology.*

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Theorem (Obstructions)

Let $N > 0$. A deformation of order N given by $m_t = \sum_{i=0}^{N} t^i m_i(x, y)$ can be extended to a deformation of order $N + 1$ if and only if the 3-cocycle obs_N is a 3-coboundary, with

$$obs_N(x, y, z) = \sum_{\substack{i+j=N \\ i,j>0}} m_i(x, m_j(y, z)) - m_i(m_j(x, y), z) - (-1)^{|x||y|} m_i(y, m_j(x, z)).$$

Positive characteristic - restricted Lie algebras

Let $\mathbb{F}$ a field of characteristic $p > 2$ and A an associative $\mathbb{F}$-algebra. With the commutator, it's a Lie algebra. The adjoint representation is then given by

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Then, if $m = p$, we obtain

$$\mathrm{ad}_x^p(y) = x^p y - y x^p = \mathrm{ad}_{x^p}(y).$$

Positive characteristic - the p-mappings

Definition (Jacobson)

A **restricted Lie algebra** *is a Lie algebra L equipped with a map $(\cdot)^{[p]} : L \longrightarrow L$ satisfying*

1. $(\lambda x)^{[p]} = \lambda^p x^{[p]}$, $x \in L$, $\lambda \in \mathbb{F}$;

Nathan Jacobson (1910-1999)

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3. $(x + y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x, y)$,

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with is$_i(x, y)$ the coefficient of Z^{i-1} in $\mathrm{ad}^{p-1}_{Zx+y}(x)$. Such a map $(-)^{[p]} : L \longrightarrow L$ is called p-map.

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Restricted module: A L-module M is called *restricted* if

$$x^{[p]} \cdot m = \Big(\overbrace{x \cdot (x \cdots (x}^{p \text{ terms}} \cdot m) \cdots)\Big), \ \forall x \in L, \ m \in M.$$

An example

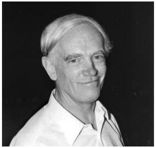
An example: the Witt algebra $W(1)$. Let char $(\mathbb{F}) = p \geq 5$. We define

$$W(1) = \text{Span}\{e_{-1}, e_0, \cdots, e_{p-2}\}$$

endowed with the bracket

$$[e_i, e_j] = \begin{cases} (j-i)e_{i+j} & \text{if } \; i+j \in \{-1, \cdots, p-2\}; \\ 0 & \text{otherwise}; \end{cases}$$

and the p-map $e_i^{[p]} = \begin{cases} e_0^{[p]} = e_0; \\ e_i^{[p]} = 0 \;\; \text{if } i \neq 0. \end{cases}$



Ernst Witt (1911-1991)

Restricted cohomology of restricted Lie algebras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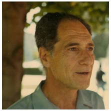
Gerhard Hochschild

Definition (Restricted 2-cochains; Evans, Fuchs)

Let $\varphi \in C^2_{CE}(L, M)$ *(ordinary Chevalley-Eilenberg 2-cochain) and* $\omega : L \longrightarrow M$*.*
Then ω **has the** $(*)$**-property w.r.t** φ *if*

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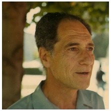
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$$\sum_{\substack{x_i = x \text{ or } y \\ x_1 = x, \ x_2 = y}} \frac{1}{\pi(x)} \sum_{k=0}^{p-2} (-1)^k x_p \dots x_{p-k+1} \varphi([[\dots[x_1, x_2], x_3] \dots, x_{p-k-1}], x_{p-k}),$$

with $x, y \in L$, $\pi(x)$ the number of factors x_i equal to x.

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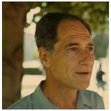
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with $x, y \in L$, $\pi(x)$ the number of factors x_i equal to x.

$$\mathfrak{C}^2(L, M) = \left\{(\varphi, \omega), \ \varphi \in C^2_{CE}(L, M), \ \omega \text{ has the } (*)\text{-property w.r.t } \varphi\right\}$$

Restricted cohomology of restricted Lie algebras

- A **restricted 2-cocycle** is an element $(\alpha, \beta) \in \mathfrak{C}^2(L, M)$ such that
  1. α is an ordinary Chevalley-Eilenberg 2-cocycle;
  2. $\alpha\left(x, y^{[p]}\right) - \sum_{i+j=p-1} (-1)^i y^i \alpha\left([x, \underbrace{y, \cdots, y}_{j \text{ terms}}], y\right) + x\beta(y) = 0.$

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- A **restricted 2-coboundary** is an element $(\alpha, \beta) \in \mathfrak{C}^2(L, M)$ such that $\exists \varphi \in \mathrm{Hom}(L, M)$,
  1. $\alpha(x, y) = \varphi([x, y]) - x\varphi(y) + y\varphi(x)$;
  2. $\beta(x) = \varphi\left(x^{[p]}\right) - x^{p-1}\varphi(x).$

We are in the following situation:

$$0 \longrightarrow \mathfrak{C}^0(L, M) \xrightarrow{d_*^0} \mathfrak{C}^1(L, M) \xrightarrow{d_*^1} \mathfrak{C}^2(L, M) \xrightarrow{d_*^2} \mathfrak{C}^3(L, M)$$

with $d_*^0 = d_{CE}^0$.

Example: Heisenberg algebras, $p > 2$

Definition (Heisenberg algebra)

The three dimensional Heisenberg algebra $\mathcal{H}$ is spanned by elements x, y, z and equipped with the Lie bracket $[\cdot,\cdot]$ defined by

$$[x, y] = z, \ \ [x, z] = [y, z] = 0.$$

Werner Heisenberg (1901-1976)

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Proposition

There are up to isomorphism three restricted Heisenberg algebras given by

1. $x^{[p]} = y^{[p]} = z^{[p]} = 0$, *denoted by* $(\mathcal{H}, 0)$;
2. $x^{[p]} = z, \ y^{[p]} = z^{[p]} = 0$, *denoted by* $(\mathcal{H}, x^*)$;
3. $x^{[p]} = y^{[p]} = 0, \ z^{[p]} = z$, *denoted by* $(\mathcal{H}, z^*)$;

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For all $u, v \in \mathcal{H}$, we have $(u + v)^{[p]} = (u)^{[p]} + (v)^{[p]}$.

Restricted cohomology with adjoint coefficients

Theorem (Second cohomology group with adjoint coefficients, $p > 3$)

We have $\dim_{\mathbb{F}}\left(H^2(\mathcal{H},0)\right) = 8$ *and* $\dim_{\mathbb{F}}\left(H^2(\mathcal{H},x^*)\right) = \dim_{\mathbb{F}}\left(H^2(\mathcal{H},z^*)\right) = 4$.

- *A basis for* $H^2(\mathcal{H},0)$ *is given by* $\{(\varphi_1,0),(\varphi_2,0),(\varphi_3,0),(\varphi_4,0),(\varphi_5,0),(0,\omega_1),(0,\omega_2),(0,\omega_3)\}$, *with*

$$\begin{aligned}\varphi_1(x,z) = z;\ \varphi_2(y,z) = z;\ \varphi_3(x,z) = -\varphi_3(y,z) = x;\\ \varphi_4(x,z) = y;\ \varphi_5(y,z) = y;\\ \omega_1(x) = z;\ \omega_2(y) = z;\ \omega_3(z) = z.\end{aligned}$$

- *A basis for* $H^2(\mathcal{H},x^*)$ *is given by* $\{(\varphi_1,0),(\varphi_2,0),(0,\omega_1),(0,\omega_2)\}$, *with*

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- *A basis for* $H^2(\mathcal{H},z^*)$ *is given by* $\{(\varphi_1,0),(\varphi_2,0),(0,\omega_1),(0,\omega_2)\}$, *with*

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- If $D \in \mathrm{Der}(A)$ and $a \in A$, we have (Hochschild)

$$(aD)^p = a^p D^p + (aD)^{p-1}(a)D.$$

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Definition

Let A be an associative commutative algebra and L a Lie algebra over a field $\mathbb{F}$ of characteristic p. Then (A, L) is a **restricted Lie-Rinehart algebra** *if*

1. *(A, L) is a Lie-Rinehart algebra, with anchor map $\rho : L \longrightarrow \mathrm{Der}(A)$;*
2. *$\left(L, (\cdot)^{[p]}\right)$ is a restricted Lie algebra;*
3. $\rho(x^{[p]}) = \rho(x)^p$;
4. $(ax)^{[p]} = a^p x^{[p]} + \rho(ax)^{p-1}(a)x, \; a \in A, \; x \in L.$

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Example: The Witt algebra with $A = \mathbb{F}[x]/(x^p - 1)$.

Definition

Let (A, L, ρ) be a restricted Lie-Rinehart algebra.
A **restricted multiderivation** *(of order 1) is a pair (m, ω), where $m : L \times L \to L$ is skew-symmetric, ω is p-homogeneous and satisfies*

$$\omega(x+y) = \omega(x) + \omega(y) + \sum_{i=1}^{p-1} \theta_i(x, y), \tag{1}$$

where $i\theta_i(x, y)$ is the coefficient of Z^{i-1} in $\left(\tilde{\mathrm{ad}}_m(Zx + y)\right)^{p-1}(x)$, with $\tilde{\mathrm{ad}}_m(x)(y) := m(x, y)$, such that it exists a map $\sigma_m : L \to \mathrm{Der}(A)$ called **restricted symbol map** *which must satisfy the following four conditions, for $x, y \in L$ and $a \in A$:*

$$\sigma(ax) = a\sigma(x); \tag{2}$$

$$m(x, ay) = am(x, y) + \sigma(x)(a)y; \tag{3}$$

$$\sigma \circ \omega(x) = \sigma(x)^p; \tag{4}$$

$$\omega(ax) = a^p \omega(x) + \sigma(ax)^{p-1}(a)x. \tag{5}$$

Proposition

Let A be an associative commutative algebra and L a Lie algebra.

There is a one-to-one correspondence between restricted Lie-Rinehart algebras structures on the pair (A, L) and restricted multiderivations of order 1 such that ($\forall x, y \in L$)

$$m(x, m(y, z)) + m(y, m(z, x)) + m(z, m(x, y)) = 0 \tag{6}$$

and

$$m(x, \omega(y)) = m(m(\ldots m(x, \overbrace{y), y), \ldots, y}^{p \text{ terms}}) \tag{7}$$

Restricted Formal Deformations

Definition

A formal deformation of (m, ω) is given, for $x, y \in L$, by two applications

$$m_t : (x,y) \longmapsto \sum_{i \geq 0} t^i m_i(x,y), \quad \omega_t : x \longmapsto \sum_{j \geq 0} t^j \omega_j(x),$$

with $m_0 = m$, $\omega_0 = \omega$, and (m_i, ω_i) restricted multiderivations. Moreover, the four following conditions must be satisfied, for $x, y, z \in L$, and $a \in A$:

$$m_t(x, m_t(y,z))_t + m_t(y, m_t(z,x)) + m_t(z, m_t(x,y)) = 0; \tag{8}$$

$$m_t\left(x, \omega_t(y)\right)_t = m_t(m_t(\cdots m_t(x, \overbrace{y), y), \cdots, y}^{p \text{ terms}}); \tag{9}$$

$$\sum_{i=0}^{k} \sigma_i\left(\omega_{k-i}(x)\right)(a) = \sum_{i_1+\cdots+i_p=k} \sigma_{i_1}(x) \circ \cdots \circ \sigma_{i_p}(x)(a), \quad \forall k \geq 0; \tag{10}$$

$$\sigma_k(x)^{p-1} = \sum_{i_1+\cdots+i_{p-1}=k} \sigma_{i_1}(x) \circ \sigma_{i_2}(x) \circ \cdots \sigma_{i_{p-1}}(x) \quad \forall k \geq 0; \tag{11}$$

Restricted Formal Deformations

Theorem

- *Let* (m_t, ω_t) *be a restricted deformation of* (m, ω)*. Then* (m_1, ω_1) *is a 2-cocyle of the restricted cohomology.*

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- *Let (m_t, ω_t) be a restricted deformation of (m, ω). Then (m_1, ω_1) is a 2-cocyle of the restricted cohomology.*
- *Let (m_t, ω_t) and (m'_t, ω'_t) be two equivalent formal deformations of (m, ω). Then, their infinitesimal elements are in the same cohomological class.*

Characteristic $p = 2$

Definition

A **restricted Lie algebra** *in characteristic $p = 2$ is a Lie algebra L endowed with a map $(\cdot)^{[2]} : L \longrightarrow L$ satisfying*

1. $(\lambda x)^{[2]} = \lambda^2 x^{[2]}$, $x \in L$, $\lambda \in \mathbb{F}$;
2. $\left[x, y^{[2]}\right] = [[x, y], y]$
3. $(x + y)^{[2]} = x^{[2]} + y^{[2]} + [x, y]$.

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3. $(x + y)^{[2]} = x^{[2]} + y^{[2]} + [x, y]$.

$\rightsquigarrow$ the third relation gives a key to understand the cohomology.

Characteristic $p = 2$

A pair (φ, ω) with $\varphi : \wedge^n L \longrightarrow M$ and $\omega : L^{n-1} \longrightarrow M$ is a n-cochain if

1. $\omega(\lambda x, z_2, \cdots, z_{n-1}) = \lambda^2 \omega(x, z_2, \cdots, z_n), \ \lambda \in \mathbb{F};$

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2. ω is multilinear in $z_2, \cdots, z_{n-1}$;
3. $\omega(x + y, z_2, \cdots, z_{n-1}) = \omega(x, z_2, \cdots, z_{n-1}) + \omega(y, z_2, \cdots, z_{n-1}) + \varphi(x, y, z_2, \cdots, z_{n-1}).$

We denote the spaces thus obtained by $\mathfrak{C}_2^n(L, M)$.

Characteristic $p = 2$

We build the differential maps $d^n_{*_2} : \mathfrak{C}^n_2(L, M) \longrightarrow \mathfrak{C}^{n+1}_2(L, M)$.

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Let $d^n_{*_2}(\varphi, \omega) = (d^n_{CE}(\varphi), \delta^n(\omega))$, with

$$\begin{aligned}\delta^n\omega(x, z_2, \cdots, z_n) &= x \cdot \varphi(x, z_2, \cdots, z_n)\\ &+ \sum_{i=2}^{n} z_i \cdot \omega(x, z_2, ..., \hat{z}_i, ..., z_n)\\ &+ \varphi(x^{[2]}, z_2, \cdots, z_n)\\ &+ \sum_{i=2}^{n} \varphi\left([x, z_i], x, z_2, \cdots, \hat{z}_i, \cdots, z_n\right)\\ &+ \sum_{1\leq i<j\leq n} \omega\left(x, [z_i, z_j], z_2, \cdots, \hat{z}_i, \cdots, \hat{z}_i, \cdots, z_n\right).\end{aligned}$$

Characteristic $p = 2$

Then :

Proposition

1. *Let* $(\varphi, \omega) \in \mathfrak{C}_2^n(L, M)$. *Then* $(d_{CE}^n(\varphi), \delta^n(\omega)) \in \mathfrak{C}_2^{n+1}(L, M)$;

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2. $\delta^{n+1} \circ \delta^n = 0$.

$\rightsquigarrow$ We can build a new full cochains complex for $p = 2$. With this complex, we obtain the same deformation results as those obtained with the Evans-Fuchs formulas for $p > 2$.

Research perspectives and ongoing works

- Classification of restricted nilpotent Lie superalgebras (Bouarroudj, Makhlouf);
- Lie-Rinehart superalgebras in charactersitic $p = 2$;
- Representations of restricted Lie-Rinehart algebras (Futorny, Makhlouf);
- Nijenhuis-Richardson algebra for restricted Lie algebras.

Publications

Journal papers

- Q. Ehret, A. Makhlouf, *On Deformations and Classification of Lie-Rinehart Superalgebras*, Communications in Mathematics **30** (2022), no. 2, 67–92.
- S. Bouarroudj, Q. Ehret, Y. Maeda, *Symplectic double extensions for restricted quasi-Frobenius Lie (super)algebras*, `arXiv:2301.12385`, accepted in SIGMA, Special Issue on Differential Geometry Inspired by Mathematical Physics in honor of Jean-Pierre Bourguignon for his 75th birthday.

Preprints

- Q. Ehret, A. Makhlouf, *Deformations and Cohomology of restricted Lie-Rinehart algebras in positive characteristic*, `arXiv:2305.16425v1`.
- Q. Ehret, A. Hajjaji, S. Mabrouk, A. Makhlouf, *On Leibniz-Rinehart Superalgebras*, in preparation.

Last Slide of the Day

Thank you for your attention!

Merci pour votre attention!