# Choosing the Best Interpolation Data in Images with Noise

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## ▶ Introduction

**Problem**

For $f, u_0 \in H^1(D)$, $\Delta f \in L^2(D)$ given and for $t > 0$, find $u(t,\cdot)$ in $H^1(D)$ such that

$$\begin{cases} \partial_t u(t,\cdot) - \Delta u(t,\cdot) = 0, \text{ in } D \setminus K_t, \\ u(t,\cdot) = f, \text{ in } K_t, \\ \frac{\partial u(t,\cdot)}{\partial \mathbf{n}} = 0, \text{ on } \partial D, \\ u(0,\cdot) = u_0, \text{ in } D. \end{cases} \quad (1)$$

The question is to identify the region $K_t$ which gives the "best" approximation $u_K$. As we want to take into account noisy images a better choice a priori is to minimize the $L^2$-norms of $u_{K_t} - f$ and its gradient [2]. Looking for the set $K_n$ at a given iteration $n$, we are led to consider the elliptic equation,

**Problem**

For $\alpha > 0$, knowning $u^n$ in $H^1(D)$, find $u$ in $H^1(D)$ such that

$$\begin{cases} u^{n+1} - \alpha \Delta u^{n+1} = u^n, \text{ in } D \setminus K_n, \\ u^{n+1} = f, \text{ in } K_n, \\ \frac{\partial u^{n+1}}{\partial \mathbf{n}} = 0, \text{ on } \partial D, \\ u^0 = u_0, \text{ in } D. \end{cases} \quad (2)$$

## ▶ The Continuous Model

Let $D$ be a bounded open subset of $\mathbb{R}^2$. We consider the shape optimization problem

$$\min_{\substack{K_n \subseteq D, \\ \mathrm{cap}(K_n) \le c}} \left\{ \frac{1}{2} \int_D |u_{K_n} - f|^2 \, dx + \frac{\alpha}{2} \int_D |\nabla (u_{K_n} - f)|^2 \, dx \;\middle|\; u_{K_n} \text{ solution of (2)} \right\}, \quad (3)$$

where cap is the capacity [3], and $c > 0$. It is well known that such shape optimization problems do not always have a solution (e.g. [1]), we seek a relaxed formulation. Thus, we consider the problem,

$$\max_{\substack{\mu_n \in \mathcal{M}_0(D), \\ \mathrm{cap}(\mu_n) < c}} \min_{u \in H^1(D)} \frac{\alpha}{2} \int_D |\nabla u|^2 \, dx + \frac{1}{2} \int_D (u - u^n)^2 \, dx + \frac{1}{2} \int_D (u - f)^2 \, d\mu_n, \quad (4)$$

where $\mathcal{M}_0(D)$ is a particular subset of all non negative Borel measures on $D$ [3, 2].

**Theorem**

(4) is the relaxed formulation of (3) in the sens of the $\gamma$-convergence.

## ▶ Topological Gradient

Here, we aim to compute the solution of our optimization problem (3) by using a topological gradient-based algorithm as in [4]. We define $K_\varepsilon^n := K_n \setminus B(x_0, \varepsilon)$ where $B(x_0, \varepsilon) \subset K_n$ and consider the functional,

$$j : A \subset D \mapsto \min_{u \in H^1(D), u = f \text{ in } A} \frac{\alpha}{2} \int_D |\nabla u|^2 \, dx + \frac{1}{2} \int_D (u - u^n)^2 \, dx.$$

**Proposition**

When $\varepsilon$ tends to 0, we have

$$j(K_\varepsilon^n) - j(K^n) = \frac{\pi}{2} \big( u^n(x_0) - f(x_0) + \alpha \Delta f(x_0) \big)^2 \varepsilon^2 \ln(\varepsilon) + O(\varepsilon^2).$$

The result above suggests to keep the points $x_0$ where $|u^n(x_0) - f(x_0) + \alpha \Delta f(x_0)|^2$ is maximal, when $\varepsilon$ is small enough.

## ▶ Optimal Distribution of Pixels

For $m > 0$ and $k \in \mathbb{N}$, we define

$$\mathcal{A}_{m,k} := \left\{ \overline{D} \cap \bigcup_{i=1}^{k} \overline{B(x_i, r)} \;\middle|\; x_i \in D_r, \; r = mk^{-1/2} \right\},$$

where $D_r$ is the $r$-neighbourhood of $D$. We consider problem (3) for every $K \in \mathcal{A}_{m,k}$ and notice that it can be reformulated as a compliance problem. We define the probability measure $\mu_K$ for a given set $K$ in $\mathcal{A}_{m,k}$ by

$$\mu_K := \frac{1}{k} \sum_{i=1}^{k} \delta_{x_i}.$$

We define the functional $F_k^n$ from $\mathcal{P}(\bar{D})$ into $[0, +\infty]$ by

$$F_k^n(\mu) := \begin{cases} k \int_D g^n v_{K_n} \, dx & , \text{ if } \exists K_n \in \mathcal{A}_{m,k}, \text{ s.t. } \mu = \mu_{K_n}, \\ +\infty & , \text{ otherwise.} \end{cases}$$

where $g^n := u^n - f + \alpha \Delta f$.

**Proposition**

The sequence of functionals $F_k^n$ $\Gamma$-converge when $k$ tends to $+\infty$ with respect to the weak $\star$ topology in $\mathcal{P}(\bar{D})$ to

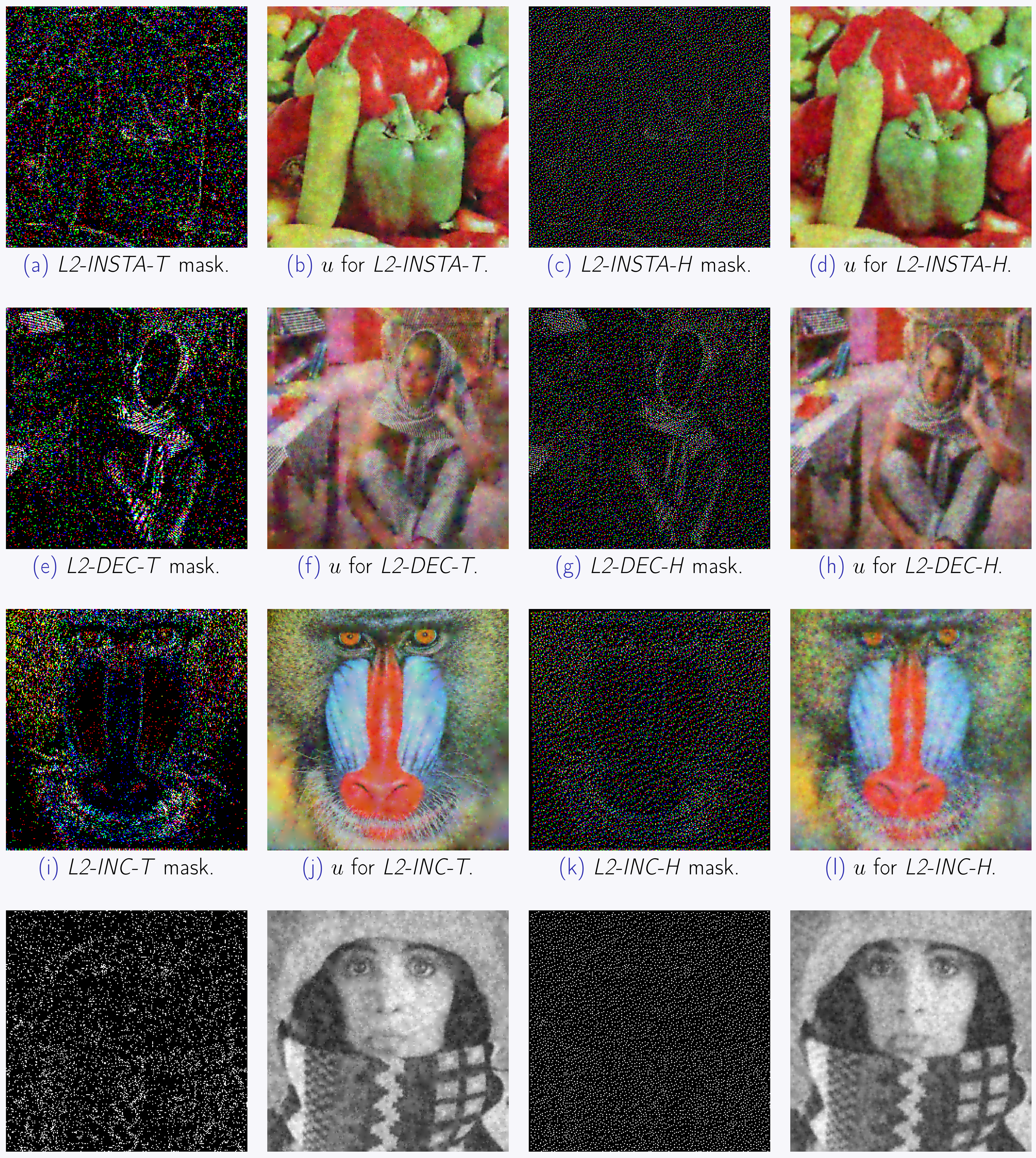
$$F^n(\mu^n) := \int_D \frac{(g^n)^2}{\mu_a^n} \theta_{K_n}\big(m(\mu_a^n)^{1/2}\big) \, dx, \quad (5)$$

where $\mu^n = \mu_a^n dx + \nu^n$ is the Radon-Nikodym-Lebesgue decomposition of $\mu^n$ with respect to the Lebesgue measure and $\theta_{K_n} : \mathbb{R} \to \mathbb{R}$.

Euler-Lagrange equation and estimates on $\theta_{K_n}$ give the following information : to minimize (5), one have to take the interpolation data such that the pixel density is increasing with $|u^n - f + \alpha \Delta f|$.

## ▶ Numerical Results

Our input image $f$ is affected by gaussian noise of standard deviation $\sigma = 0.05$ to become $f_\sigma$. For the encoding step, we take $u_0 = f_\sigma$ and save 10% of total pixel in the inpainting mask, while for the decoding step we use $u_0 = 0$. Used algorithms are described in [2].



(a) L2-INSTA-T mask. (b) $u$ for L2-INSTA-T. (c) L2-INSTA-H mask. (d) $u$ for L2-INSTA-H.
(e) L2-DEC-T mask. (f) $u$ for L2-DEC-T. (g) L2-DEC-H mask. (h) $u$ for L2-DEC-H.
(i) L2-INC-T mask. (j) $u$ for L2-INC-T. (k) L2-INC-H mask. (l) $u$ for L2-INC-H.
(m) L2-STA-T mask. (n) $u$ for L2-STA-T. (o) L2-STA-H mask. (p) $u$ for L2-STA-H.

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