Optimal Interpolation Data for PDE-Based Compression of Images with Noise

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Outline

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Goal : reconstruct missing parts of an image $f : D \to [0,1]$ from a set $K \subset D$ of known pixels.

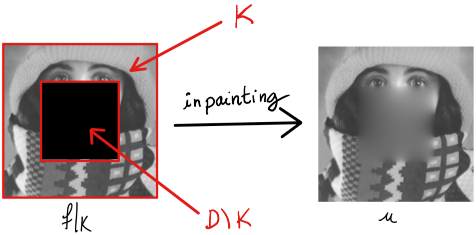

Figure – Image Inpainting

General model of PDE-based inpainting : For a given $K \subset D$,

$$\begin{cases} A(u) = 0, & \text{in } D \setminus K, \\ u = f, & \text{in } K, \\ \frac{\partial u}{\partial n} = 0, & \text{on } \partial D, \end{cases} \tag{1}$$

where

- A is the inpainting operator,
- u is the reconstructed image (inpainted image),
- f is the original image, available on K only.

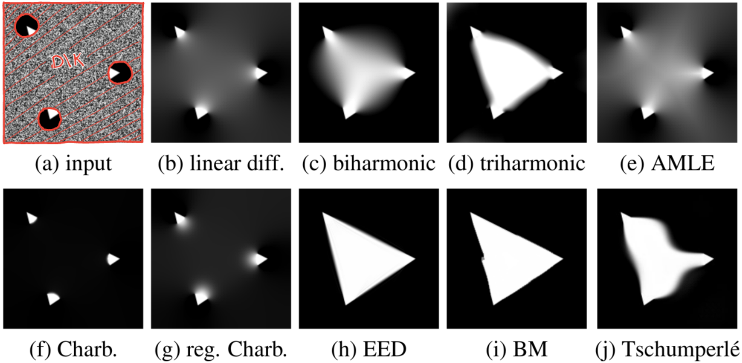

Figure – Examples.

Christian Schmaltz et al. "Understanding, Optimising, and Extending Data Compression with Anisotropic Diffusion". In : *International Journal of Computer Vision* 108 (1er juill. 2014). doi : 10.1007/s11263-014-0702-z

We remove parts of the image and reconstruct them by inpainting :

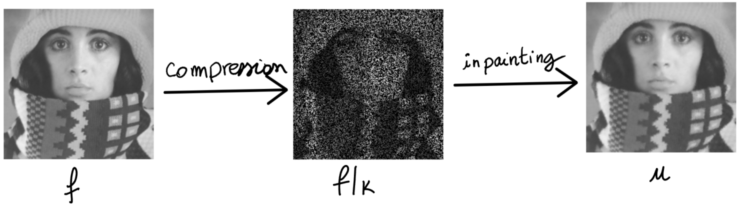

Figure – Compression by Inpainting.

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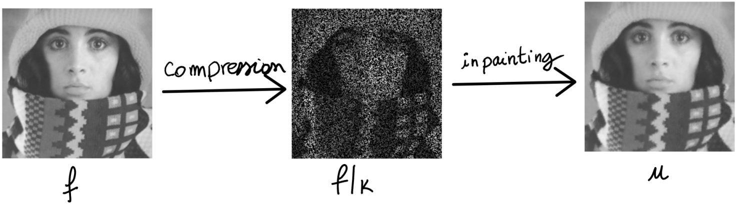

Figure – Compression by Inpainting.

The compression is lossy :

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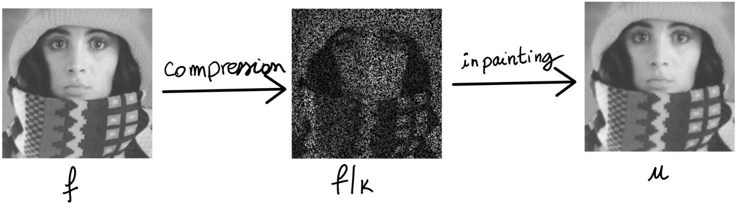

Figure – Compression by Inpainting.

The compression is lossy : for a given reconstruction method, how to choose K such that u and f are close ?

Compression problem

Find a subset K of known pixels solution of,

$$\min_{K \subseteq D,\ \mathrm{cap}(K) \leq c} \{\mathcal{E}(u_K) \mid u_K \text{ solution of an inpainting problem}\},$$

where cap is the set capacity,

$$\mathrm{cap}(K) := \inf \left\{ \int_D |\nabla u|^2 \, dx + \int_D u^2 \, dx \,\middle|\, u \in H_0^1(D),\ u = 1 \text{ a.e. in } K \right\},$$

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The existence of a solution depends on the inpainting problem and on the error $\mathcal{E}$.

If the inpainting is the homogeneous diffusion inpainting i.e.

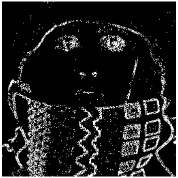
$$A(u) = \Delta u,$$

and if the error is

$$\mathcal{E}(u) = |u - f|_{H^1(D)},$$

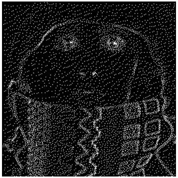
then, the compression problem admits at least a (relaxed) solution in a suitable sense.

Zakaria Belhachmi et al. "How to Choose Interpolation Data in Images". In : *SIAM Journal of Applied Mathematics* 70 (1er jan. 2009), p. 333-352. doi : 10.1137/080716396



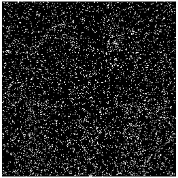
(a) K ($H1$-T).

(b) u_K ($H1$-T).



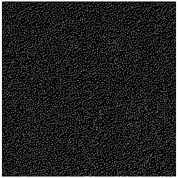
(c) K ($H1$-H).

(d) u_K ($H1$-H).



(e) K ($H1$-T).

(f) u_K ($H1$-T).



(g) K ($H1$-H).

(h) u_K ($H1$-H).

Outline

Compression problem

For a $n \in \mathbb{N}$ fixed, find a subset K_n of known pixels solution of,

$$\inf_{K_n \subseteq D,\ \mathrm{cap}(K_n) \leq c_n} \{\mathcal{E}(u_{K_n}) \mid u_{K_n} \text{ solution of } (3)\}, \tag{2}$$

with

$$\mathcal{E}(u) = \int_D |u - f|^2 \, dx + \alpha \int_D |\nabla u - \nabla f|^2 \, dx,$$

and the inpainting problem

$$\begin{cases} -\alpha \Delta u + u = u^n, & \text{in } D \setminus K_n, \\ u = f, & \text{in } K_n, \\ \frac{\partial u}{\partial n} = 0, & \text{on } \partial D. \end{cases} \tag{3}$$

Formally,

$$\begin{cases} -\alpha\Delta u + u = u^n, & \text{in } D \setminus K_n, \\ u = f, & \text{in } K_n, \\ \frac{\partial u}{\partial n} = 0, & \text{on } \partial D, \end{cases}$$

is a discretization in time of

$$\begin{cases} \partial_t u(t,\cdot) - \Delta u(t,\cdot) = 0, & \text{in } D \setminus K_t, \\ u(t,\cdot) = f, & \text{in } K_t, \\ \frac{\partial u(t,\cdot)}{\partial \mathrm{n}} = 0, & \text{on } \partial D, \end{cases}$$

with $u(0,\cdot) = u_0,$ in D, for a given u_0.

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with $u(0,\cdot) = u_0,$ in D, for a given u_0.

- We focus on a fixed iteration n for a given u^n.

Proposition

The compression problem (4) is equivalent, for $\beta > 0$, to

$$\sup_{K_n \subseteq D} \min_{u \in H^1(D)} \frac{\alpha}{2} \int_D |\nabla u|^2 \, dx + \frac{1}{2} \int_D (u - u^n)^2 \, dx + \int_D (u - f)^2 \, d\infty_{K_n} - \beta \, cap(\infty_{K_n}).$$

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For a $n \in \mathbb{N}$ fixed, find a subset K_n of known pixels solution of,

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$$F_\mu(u) := \begin{cases} \alpha \int_D |\nabla u|^2 \; dx + \int_D (u - u^n)^2 \; dx + \int_D (u - f)^2 \; d\mu, & \text{if } |u| \leq |f|_\infty, \\ +\infty, & \text{otherwise}, \end{cases}$$

$$E(\mu) := \min_{u \in H^1(D)} F_\mu(u).$$

Theorem

We have,

$$\sup_{K_n \in \mathcal{K}_\delta(D)} \Big(E(\infty_{K_n}) - \beta\,\mathrm{cap}(\infty_{K_n})\Big) = \max_{\mu_n \in \mathcal{M}_0^\delta(D)} \Big(E(\mu_n) - \beta\,\mathrm{cap}(\mu_n)\Big).$$

with, for $\delta > 0$,

$$D^{-\delta} := \{x \in D \mid d(x, \partial D) \geq \delta\} \subseteq D,$$

$$\mathcal{K}^\delta(D) := \{K \subseteq D \mid K \text{ closed}, \; K \subseteq D^{-\delta}\},$$

and

$$\mathcal{M}_0^\delta(D) := \{\mu \in \mathcal{M}_0(D) \mid \mu|_{D \setminus D^{-\delta}} = 0\} \subseteq \mathcal{M}_0(D).$$

Outline

1. The General Compression Model
2. The Model Considering Noise
3. A Construction of the Optimal Set
4. Numerical Results

Let us denote by j the functional

$$j : A \subset D \mapsto \min_{u \in H^1(D),\ u=f \text{ in } A} \frac{1}{2}\int_D (u-u^n)^2\,dx + \frac{\alpha}{2}\int_D |\nabla u|^2\,dx.$$

We have

Proposition

With notations from above, we have when ε tends to 0,

$$j(K_\varepsilon^n) - j(K_n) = \frac{\pi}{2}\big(u^n(x_0) - f(x_0) + \alpha\Delta f(x_0)\big)^2 \varepsilon^2 \ln(\varepsilon) + O(\varepsilon^2).$$

The result above suggests to keep the points x_0 where $|u^n(x_0) - f(x_0) + \alpha\Delta f(x_0)|$ is maximal, when ε small enough.

For $m > 0$ and $k \in \mathbb{N}$, we define

$$\mathcal{A}_{m,k} := \left\{ \overline{D} \cap \bigcup_{i=1}^{k} \overline{B(x_i, r)} \;\middle|\; x_i \in D_r,\ \ r = mk^{-1/2} \right\},$$

where D_r is the r-neighborhood of D.

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where D_r is the r-neighborhood of D. We consider the compression problem for every $K_n \in \mathcal{A}_{m,k}$ i.e.

$$\min_{K_n \in \mathcal{A}_{m,k}} \left\{ \mathcal{E}(u_{K_n}) \;\middle|\; u_{K_n} \text{ solution of the inpainting problem} \right\}.$$

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It can be written as a compliance problem i.e.

$$\min_{K_n \in \mathcal{A}_{m,k}} \left\{ \int_D g^n (u_{K_n} - f)\, dx \;\middle|\; u_{K_n} \text{ solution of the inpainting problem} \right\},$$

with $g^n := u^n - f + \alpha \Delta f$.

We define the functional F_k^n from $\mathcal{P}(\bar{D})$ into $[0, +\infty]$ by

$$F_k^n(\mu) := \begin{cases} k \int_D g^n(u_{K_n} - f)\ dx & , \text{ if } \exists K_n \in \mathcal{A}_{m,k}, \text{ s.t. } \mu = \mu_{K_n}, \\ +\infty & , \text{ otherwise,} \end{cases}$$

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Theorem

$(F_k^n)_k$ Γ-converge when k tends to $+\infty$ w.r.t. the weak $\star$ topology in $\mathcal{P}(\bar{D})$ to

$$F^n(\mu^n) := \int_D \frac{(g^n)^2}{\mu_a^n} \theta\left(m(\mu_a^n)^{1/2}\right)\, dx,$$

where $\mu^n := \mu_a^n dx + \nu^n$ and $\theta : \mathbb{R} \to \mathbb{R}_+$.

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Giuseppe Buttazzo, Filippo Santambrogio et Nicolas Varchon. "Asymptotics of an optimal compliance-location problem". In : *ESAIM : Control, Optimisation and Calculus of Variations* 12 (avr. 2005). doi : 10.1051/cocv:2006020

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Formally, Euler-Lagrange equation gives : to minimize (5) one have to take

$$(\mu_a^n)^2/|1 - \log \mu_a^n| \approx c_{m,f}\,(\underbrace{u^n - f + \alpha\Delta f}_{=:\ g^n})^2.$$

Outline

Data: Original (noisy) image f, parameter $\alpha > 0$, desired pixel density $c \in (0, 1)$, number of iterations $N \in \mathbb{N}^*$.
Result: Inpainting mask $K \subset D$, last reconstruction u^N.

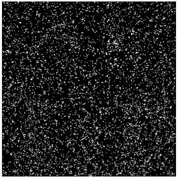
$u^0 \leftarrow f$;
for n *in* $\{0, \ldots, N-1\}$ **do**
  Save in K the $c\,|D|$ pixel by using $|u^n - f + \alpha \Delta f|$;
  Compute u^{n+1}, solution of the inpainting problem;
end

Algorithm 1: *L2-INSTA*.

Data: Original (noisy) image f, parameter $\alpha > 0$, fraction q of pixel added, desired pixel density $c \in (0, 1)$.
Result: Inpainting mask $K \subset D$, last reconstruction u^N.

$u^0 \leftarrow f$;
$K \leftarrow \emptyset$;
$n \leftarrow 0$;
while $|K| < c\,|D|$ **do**
  Add the $q\,|D|$ pixels in $D \setminus K$ to K by using $|u^n - f + \alpha \Delta f|$;
  Compute u^{n+1}, solution of the inpainting problem;
  $n \leftarrow n + 1$;
end

Algorithm 2: *L2-INC.*



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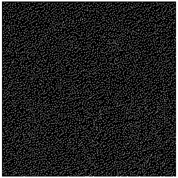
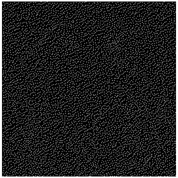
(j) u_K (*H1-T*).



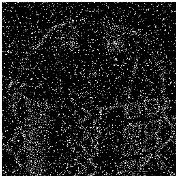
(k) K (*H1-H*).

(l) u_K (*H1-H*).



(m) K (*L2-INSTA-H*).

(n) u_K (*L2-INSTA-H*).



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$$\min_{K_n \subseteq D,\ \mathrm{cap}(K_n) \leq c} \left\{ \frac{1}{2} \int_D |u_{K_n} - u^n|^2 \, dx + \frac{\alpha}{2} \int_D |\nabla (u_{K_n} - u^n)|^2 \, dx \right\},$$

with u_{K_n} solution of

$$\begin{cases} u - \alpha \Delta u = u^n, & \text{in } D \setminus K_n, \\ u = u^n, & \text{in } K_n, \\ \frac{\partial u}{\partial \mathrm{n}} = 0, & \text{on } \partial D. \end{cases}$$

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- Previous criterion : $|u^n - f + \alpha \Delta f|$,

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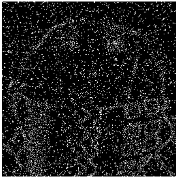
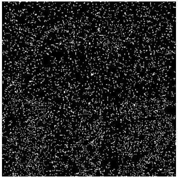
  - Previous criterion : $|u^n - f + \alpha \Delta f|$,
- New criterion : $|u^n - u^n + \alpha \Delta u^n| = \alpha |\Delta u^n|$.



(q) K ($L2$-INC).

(r) u_K ($L2$-INC).

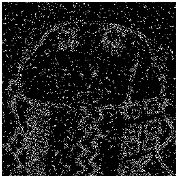


(s) K ($L2$-INC).

(t) u_K ($L2$-INC).

(u) K ($L2$-INC-E).

(v) u_K ($L2$-INC-E).



(w) K ($L2$-INC-E).

(x) u_K ($L2$-INC-E).

Thanks for your attention !