PDE-Based Image Compression

Thomas Jacumin (joint work with Zakaria Belhachmi)

January 29, 2026

Organization

1. JPEG
2. Modeling
3. Analysis
4. Numerical

- An image is divided into blocks of size 8×8 pixels

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- F is decomposed:

$$F = \sum_i c_i \Phi_i.$$

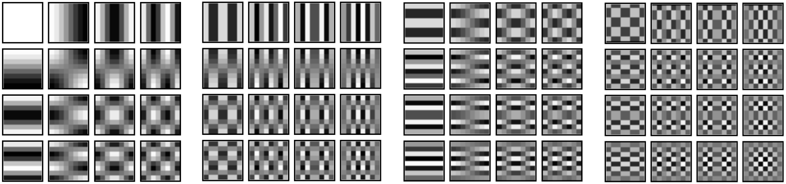

Figure: The 64 base images Φ_i for the JPEG standard.

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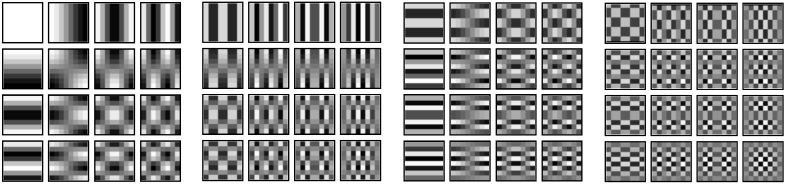

Figure: The 64 base images Φ_i for the JPEG standard.

- We save the c_i for each block.

Problem

Presence of visual artifacts in the image.

Original | JPEG | JPEG2000 | WEBP

Figure: Appearance of visual artifacts at high compression rates.

Organization

Goal: Reconstruct a missing part of an image $f : K \subseteq D \to [0,1]$ given.

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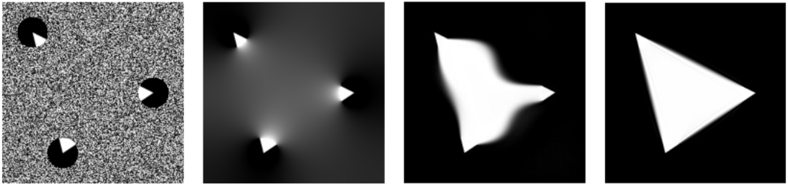
Inpainting

Find $u : D \to [0,1]$ such that :

$$\begin{cases} A(u) = 0 & \text{in } D \setminus K, \\ u = f & \text{in } K, \\ \frac{\partial u}{\partial \mathrm{n}} = 0 & \text{on } \partial D, \end{cases}$$

with D being the support of the image, and A an operator (PDE).



Heat equation

Figure: Some examples of image inpainting.

Heat equation:

$$\begin{cases} \partial_t u = \Delta u & \text{in } [0, +\infty[\times D \setminus K, \\ u = f & \text{in } [0, +\infty[\times K, \\ \frac{\partial u}{\partial \mathrm{n}} = 0 & \text{on } [0, +\infty[\times \partial D, \end{cases}$$

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Time discretization: $\partial_t u \approx \frac{u_{n+1} - u_n}{\alpha}$,

$$\begin{cases} u - \alpha \Delta u = u_0 & \text{in } D \setminus K, \\ u = f & \text{in } K, \\ \frac{\partial u}{\partial \mathrm{n}} = 0 & \text{on } \partial D. \end{cases}$$

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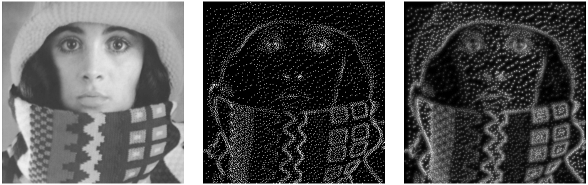

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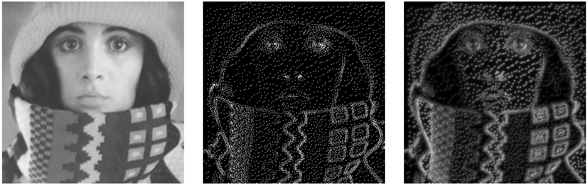
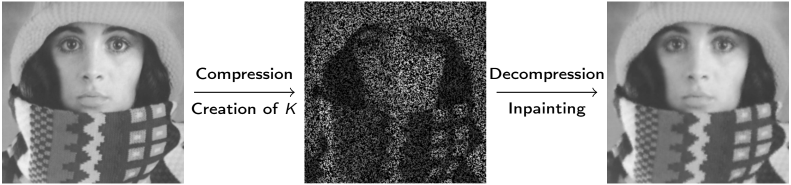


Applications of inpainting:

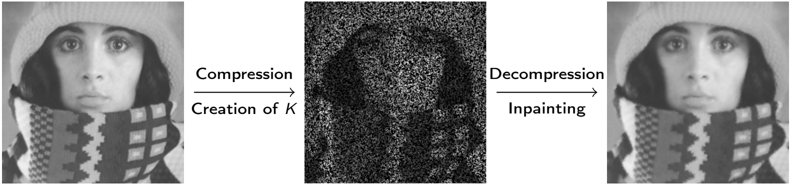
- Restoration,
- Removal of unwanted elements (this is what interests us).



Original image f. Compressed image. Decompressed image u.

Figure: Compression by inpainting.



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Question

Which pixels to remove?

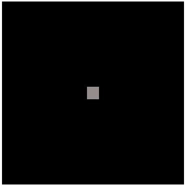
Shape optimization problem:

$$\min_{K \subseteq D,\ m(K) \leq c} \left\{ \mathcal{E}(u_K) \;\middle|\; u_K \text{ solution of the inpainting} \right\},$$

with

- $\mathcal{E}$ the error between the decompressed image and the original image,
- m quantifies the size of K.

Example of the importance of the choice of K: Only one pixel.



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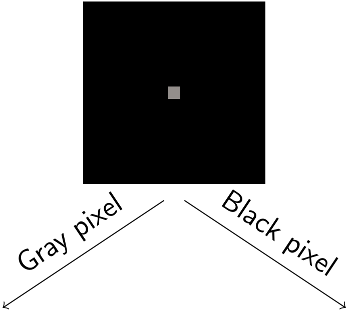

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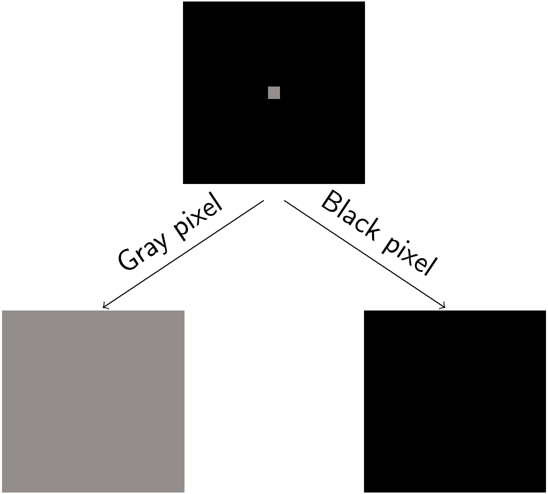

$\mathcal{E}(u_K)$ large.

$\mathcal{E}(u_K)$ small + denoising.

If we consider that the image is noisy, we can remove this noise by compressing.

Original — Impulse noise — Gaussian noise

Figure: Example of noise.

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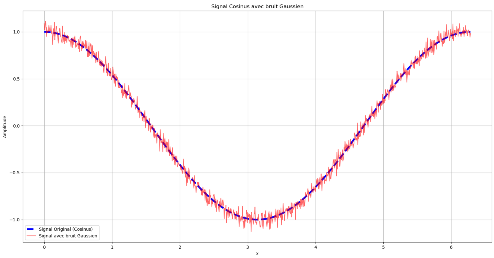
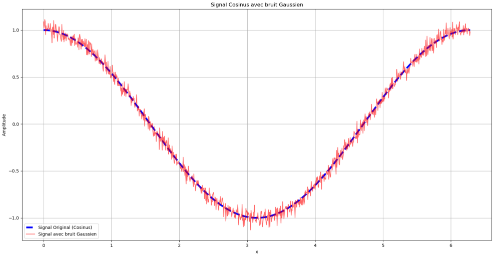
Classic in image processing: for $\alpha > 0$,

$$\min_u \{\mathcal{D}(u, f) + \alpha \mathcal{R}(u)\}.$$

Gaussian: we want a regular image close to f, but without penalizing small deviations too much.

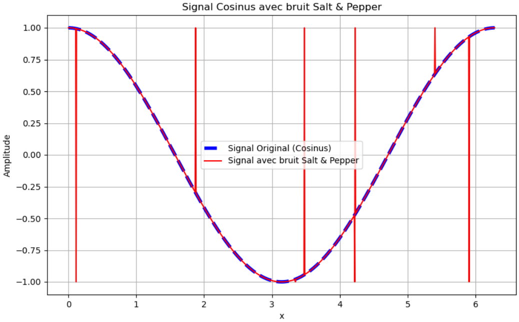


Gaussian: we want a regular image close to f, but without penalizing small deviations too much.



$$\min_u \{ \frac{1}{2} \int_D (u-f)^2 \, dx + \alpha \mathcal{R}(u) \}.$$

Impulse noise: we want a regular image close to f, but without penalizing large deviations too much.



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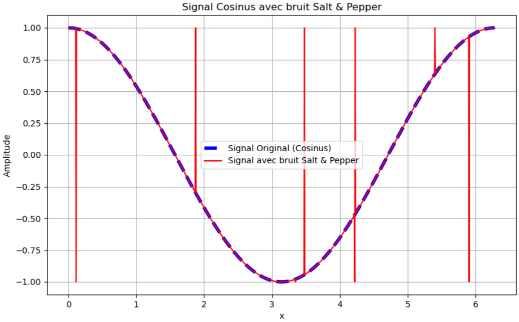

$$\min_u \{ \int_D |u - f| \ dx + \alpha \mathcal{R}(u) \}.$$

Compression problem

For $1 \leq p \leq 2$,

$$\min_{K \subseteq D,\ m(K) \leq c} \left\{ \frac{1}{p} \int_D |u_K - f|^p \, dx \;\middle|\; u_K \text{ solution of } (1) \right\},$$

where

$$\begin{cases} u - \alpha \Delta u = u_0 & \text{in } D \setminus K, \\ u = f & \text{in } K, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial D. \end{cases} \tag{1}$$

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- no $\mathcal{R}$,

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What remains is to determine m.

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$\underline{m = \mathcal{L}}$: Problem to penalize lines and points.

Example:

$$\min_{K \subseteq D,\ \mathcal{L}(K) \leq c} \underbrace{\int_{D \setminus K} |\nabla u_K|^2 \, dx}_{=:\mathcal{J}(u_K)},$$

where

$$\begin{cases} -\Delta u_K + u_K = 1 & \text{in } D \setminus K, \\ u_K = 0 & \text{in } K, \\ u_K = 0 & \text{on } \partial D. \end{cases}$$

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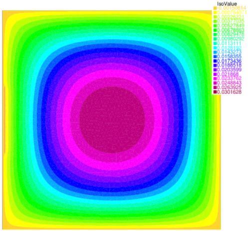

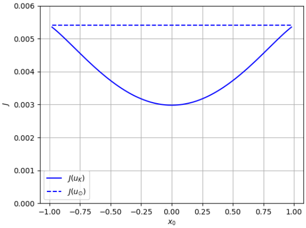

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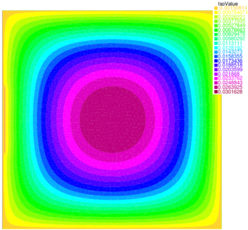

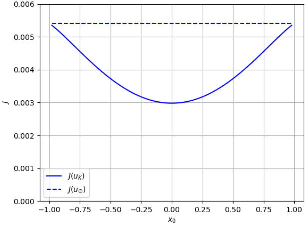

$\underline{m = \mathcal{H}^1}$: Problem to penalize points.

Example:

$$\min_{K \subseteq D,\ \mathcal{H}^1(K) \leq c} \underbrace{\int_{D \setminus K} |\nabla u_K|^2 \, dx}_{=:\mathcal{J}(u_K)},$$

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We can construct a K such that

- $\mathcal{H}^1(K) = 0$,
- $\mathcal{J}(u_K) \leq \mathcal{J}(u_\emptyset)$.

We need m:

- proportional to the size of K,
- quantifies how K influences the solution of the PDE.

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$m = \text{cap}$ (Capacity):

Definition (Capacity)

$$\text{cap}(K) := \inf \left\{ \int_D |\nabla u|^2 \, dx + \int_D u^2 \, dx \;\middle|\; u \in H_0^1(D), \right. \\ \left. u = 1 \text{ a.e. in } K \right\}.$$

- proportional to the size of K: if $D = B(0, R)$, for sufficiently small r,

$$\mathrm{cap}\big(B(0,r)\big) = \frac{-C}{\ln(r)}.$$

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Proposition

$cap(K) = 0$ if and only if $u_K = u_\emptyset$.

Compression problem

For $1 \leq p \leq 2$,

$$\min_{K \subseteq D,\ \mathrm{cap}(K) \leq c} \left\{ \underbrace{\frac{1}{p} \int_D |u_K - f|^p \, dx}_{\mathcal{E}(u_K)} \;\middle|\; u_K \text{ solution of } (2) \right\},$$

where

$$\begin{cases} u - \alpha \Delta u = u_0 & \text{in } D \setminus K, \\ u = f & \text{in } K, \\ \frac{\partial u}{\partial \mathrm{n}} = 0 & \text{on } \partial D. \end{cases} \tag{2}$$

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Theorem

For $p = 2$, the compression problem admits a (relaxed) solution.

Question

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Definition (Topological gradient)

Let $x_0 \in D$ and $K_\varepsilon = K \cup \overline{B(x_0, \varepsilon)}$.

$$\mathcal{E}(u_{K_\varepsilon}) - \mathcal{E}(u_K) = \rho(\varepsilon)\, G(x_0) + o\big(\rho(\varepsilon)\big),$$

where

- ρ is a positive function going to zero as ε approaches zero,
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To minimize the cost functional, one has to create small holes at the locations x_0 where $G(x_0)$ is the most negative.

Setting $\widetilde{v}_K = u_K - f$, we can write equivalently,

Problem

Find $\widetilde{v}_K$ in $H^1(D)$ such that

$$\begin{cases} -\alpha\Delta\widetilde{v}_K + \widetilde{v}_K = \alpha\Delta f & \text{in } D \setminus K, \\ \widetilde{v}_K = 0 & \text{in } K, \\ \frac{\partial \widetilde{v}_K}{\partial \mathbf{n}} = 0 & \text{on } \partial D. \end{cases}$$

For the analysis: $K_\varepsilon := B(x_0, \varepsilon)$, $\widetilde{v}_\varepsilon := \widetilde{v}_{K_\varepsilon}$ and $\widetilde{v}_0 := \widetilde{v}_\emptyset$.

Weak formulation: find $\widetilde{v}_\varepsilon$ in V_ε such that,

$$a_\varepsilon(\widetilde{v}_\varepsilon, \varphi) = l_\varepsilon(\varphi), \ \forall \varphi \in V_\varepsilon,$$

with

$$\begin{aligned}
V_\varepsilon &:= \{ v \in H^1(D \setminus B_\varepsilon) \mid v = 0 \text{ on } \partial B_\varepsilon \}, \\
a_\varepsilon(\widetilde{v}_\varepsilon, \varphi) &:= \alpha \int_{D \setminus B_\varepsilon} \nabla \widetilde{v}_\varepsilon \cdot \nabla \varphi \, dx + \int_{D \setminus B_\varepsilon} \widetilde{v}_\varepsilon \, \varphi \, dx, \\
l_\varepsilon(\varphi) &:= \int_{D \setminus B_\varepsilon} h \, \varphi \, dx.
\end{aligned}$$

we suppose that there exists

- a continuous bilinear form $\delta a : V \times V \to \mathbb{R}$,
- a continuous linear form $\delta l : V \to \mathbb{R}$,
- a function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ such that, for all $\varepsilon \geq 0$,

such that

- $\|a_\varepsilon - a_0 - \rho(\varepsilon)\,\delta a\|_{\mathcal{L}_2(V)} = o\big(\rho(\varepsilon)\big)$,
- $\|l_\varepsilon - l_0 - \rho(\varepsilon)\,\delta l\|_{\mathcal{L}(V)} = o\big(\rho(\varepsilon)\big)$,
- $\lim_{\varepsilon \to 0} \rho(\varepsilon) = 0$.

Theorem

Let $\widetilde{v}_\varepsilon \in V$ be the solution of the following problem : find $\widetilde{v} \in V$ such that,

$$a_\varepsilon(\widetilde{v}, \varphi) = l_\varepsilon(\varphi), \ \forall \varphi \in V.$$

Let $\widetilde{w_0}$ be the solution of the so-called adjoint problem: find $\widetilde{w} \in V$ such that

$$a_0(\widetilde{w}, \varphi) = -D\mathcal{E}(\widetilde{v_0})\varphi = -\int_D \widetilde{v_0}|\widetilde{v_0}|^{p-2}\varphi \, dx, \ \forall \varphi \in V.$$

Then,

$$\mathcal{E}(\widetilde{v_\varepsilon}) - \mathcal{E}(\widetilde{v_0}) = \rho(\varepsilon)\big(\delta a(\widetilde{v_0}, \widetilde{w_0}) - \delta l(\widetilde{w_0})\big) + o\big(\rho(\varepsilon)\big).$$

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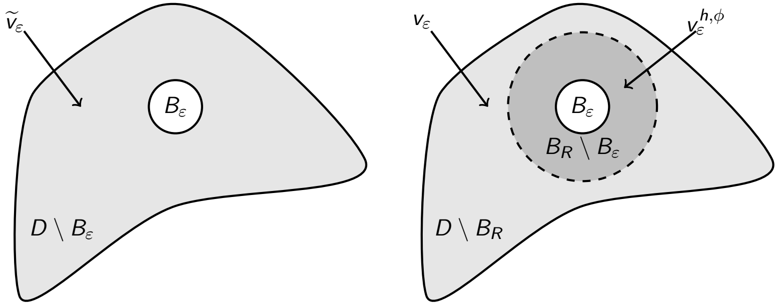
Problem: our $V = V_\varepsilon := \{v \in H^1(D \setminus B_\varepsilon) \mid v = 0 \text{ on } \partial B_\varepsilon\}$ depends on ε.

Truncation technique :

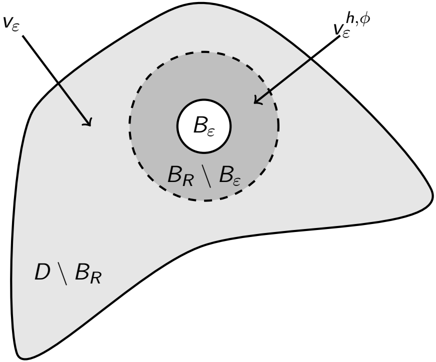


Before splitting.

After splitting.

Truncation technique : *internal* problem

$$\begin{cases} -\alpha\Delta v_\varepsilon^{h,\phi} + v_\varepsilon^{h,\phi} = h := \alpha\Delta f & \text{in } B_R \setminus B_\varepsilon, \\ v_\varepsilon^{h,\phi} = 0 & \text{on } \partial B_\varepsilon, \\ v_\varepsilon^{h,\phi} = \phi := v_\varepsilon & \text{on } \partial B_R. \end{cases}$$



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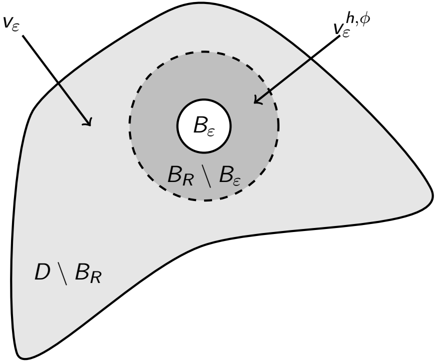

Proposition

We have

$$\widetilde{v}_\varepsilon = \begin{cases} v_\varepsilon & \text{in } D \setminus B_R, \\ v_\varepsilon^{h,\phi} & \text{in } B_R \setminus B_\varepsilon. \end{cases}$$

For v, φ in $V_R := H^1(D \setminus B_R)$, we define

$$a_\varepsilon(v, \varphi) := \alpha \int_{D\setminus B_R} \nabla v \cdot \nabla \varphi \; dx + \alpha \int_{\partial B_R} \partial_n v_\varepsilon^{0,\phi} \, \varphi \; d\sigma + \int_{D\setminus B_R} v \, \varphi \; dx,$$

$$l_\varepsilon(\varphi) := \int_{D\setminus B_R} h \, \varphi \; dx - \alpha \int_{\partial B_R} \partial_n v_\varepsilon^{h,0} \, \varphi \; d\sigma.$$

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We need to compute δa and δl.

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$$l_\varepsilon(\varphi) := \int_{D\setminus B_R} h \, \varphi \, dx - \alpha \int_{\partial B_R} \partial_n v_\varepsilon^{h,0} \varphi \, d\sigma.$$

δa and δl

$$a_\varepsilon(v,\varphi) - a_0(v,\varphi) = \alpha \int_{\partial B_R} \partial_n (v_\varepsilon^{0,\phi} - v_0^{0,\phi}) \, \varphi \, d\sigma,$$

$$l_\varepsilon(\varphi) - l_0(\varphi) = -\alpha \int_{\partial B_R} \partial_n (v_\varepsilon^{h,0} - v_0^{h,0}) \, \varphi \, d\sigma.$$

For $v_\varepsilon^{0,\phi}$:

Proof.

- Using polar coordinates in $\mathbb{R}^2$, we have,

$$v_\varepsilon^{0,\phi}(r,\theta) = \sum_{n\in\mathbb{Z}} c_n(r)\, e^{in\theta},$$

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$$v_\varepsilon^{0,\phi}(r,\theta) = \sum_{n\in\mathbb{Z}} c_n(r)\, e^{in\theta},$$

- where c_n satisfies, for all n in $\mathbb{Z}$, and $0 < r \leq R$,

$$-\alpha\, r^2\, c_n''(r) - \alpha\, r\, c_n'(r) + (r^2 + \alpha n^2) c_n(r) = 0.$$

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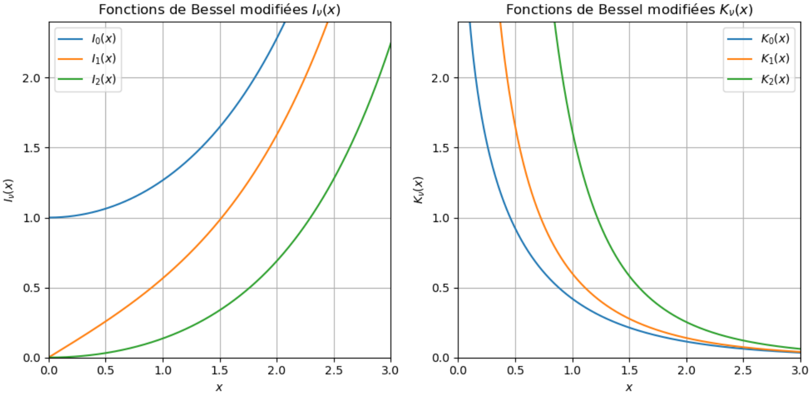
- We solve the equation using Bessel functions. □


Fonctions de Bessel modifiées $I_\nu(x)$
$I_0(x)$
$I_1(x)$
$I_2(x)$
$I_\nu(x)$
0.0
0.5
1.0
1.5
2.0
0.0
0.5
1.0
1.5
2.0
2.5
3.0
x
Fonctions de Bessel modifiées $K_\nu(x)$
$K_0(x)$
$K_1(x)$
$K_2(x)$
$K_\nu(x)$
0.0
0.5
1.0
1.5
2.0
0.0
0.5
1.0
1.5
2.0
2.5
3.0
x

For $v_\varepsilon^{h,0}$:

Proof.

- Using polar coordinates in $\mathbb{R}^2$, we have,

$$v_\varepsilon^{h,0}(r,\theta) = \sum_{n\in\mathbb{Z}} c_n(r)\, e^{in\theta} \qquad \text{and} \qquad h(r,\theta) = \sum_{n\in\mathbb{Z}} h_n(r)\, e^{in\theta},$$

- where c_n satisfies, for all n in $\mathbb{Z}$ and $0 < r \leq R$,

$$-\alpha\, r^2\, c_n''(r) - \alpha\, r\, c_n'(r) + (r^2 + \alpha n^2) c_n(r) = r^2 h_n(r).$$

- We solve the equation using Bessel functions. □

Proposition

For ε small enough, we have,

$$j(K_\varepsilon) - j(K) = \underbrace{4\pi\alpha\, \widetilde{v}_0(x_0)\, \widetilde{w}_0(x_0)}_{=G(x_0)} \frac{-1}{\ln \varepsilon} + o\left(\frac{-1}{\ln \varepsilon}\right),$$

with

$$\begin{cases} -\alpha\Delta\widetilde{v}_0 + \widetilde{v}_0 = \alpha\Delta f & \text{in } D, \\ \partial_n \widetilde{v}_0 = 0 & \text{on } \partial D, \end{cases}$$

and

$$\begin{cases} -\alpha\Delta\widetilde{w}_0 + \widetilde{w}_0 = -\widetilde{v}_0 |\widetilde{v}_0|^{p-2} & \text{in } D, \\ \partial_n \widetilde{w}_0 = 0 & \text{on } \partial D. \end{cases}$$

Question

~~Which pixels to remove?~~ Which pixels to keep?

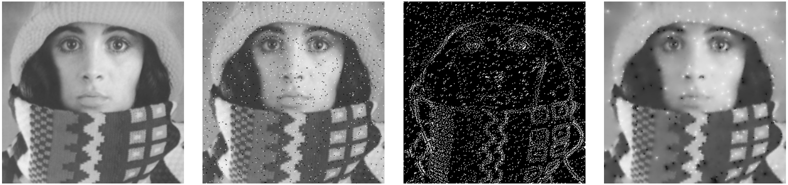
Question

~~Which pixels to remove?~~ Which pixels to keep?

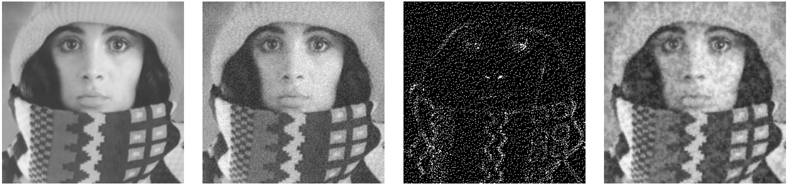
Answer

We have to keep in the mask the pixels x_0 which minimize the product $v_0(x_0)w_0(x_0)$.

Organization



Original | Noisy | K $(p = 1)$ | u_K



Original | Noisy | K $(p = 2)$ | u_K

Figure: 10% of the total number of pixels.

Conclusion:

- Solution for $p \neq 2$,
- Another operator,
- Difficulty in saving K in practice,
- Not great for Gaussian noise (there are empirical techniques).

Thank you for your attention!